IMPLOSION FOR HYPERKÄHLER MANIFOLDS

ANDREW DANCER, FRANCES KIRWAN, AND ANDREW SWANN

Abstract. We introduce an analogue in hyperkähler geometry of the symplectic implosion, in the case of $SU(n)$ actions. Our space is a stratified hyperkähler space which can be defined in terms of quiver diagrams. It also has a description as a non-reductive geometric invariant theory quotient.

Introduction

Guillemin, Jeffrey and Sjamaar [9] introduced the idea of symplectic implosion of a symplectic manifold M with a Hamiltonian action of a compact group K. The implosion M_{impl} carries an action of a maximal torus T of K, such that the symplectic reductions of M by K agree with the symplectic reductions of the implosion by T. In this sense the implosion is an abelianisation of the original Hamiltonian action; the price to be paid for this is that the implosion is usually quite singular, although it has a stratified symplectic structure.

The construction of symplectic implosions can be reduced to the problem of imploding the cotangent bundle T^*K, which thus acts as a universal implosion. The imploded space $(T^*K)_{\text{impl}}$ carries a torus action such that the symplectic reductions are the coadjoint orbits of K. The universal symplectic implosion $(T^*K)_{\text{impl}}$ also has an algebro-geometric description as the canonical affine completion of the quotient $K_{\mathbb{C}}/N$ of the complexified group $K_{\mathbb{C}}$ by a maximal unipotent subgroup N.

Our aim here is to explore a hyperkähler analogue of the universal implosion. In this paper we concentrate on the case of $SU(n)$ actions, where there is a construction involving quiver diagrams, leaving the case of other compact groups K to a future paper. We produce a stratified hyperkähler space Q whose strata correspond to quiver diagrams of suitable types. These strata are hyperkähler manifolds which can be described in terms of open sets in complex symplectic quotients of the cotangent bundle of $K_{\mathbb{C}} = SL(n, \mathbb{C})$ by subgroups containing commutators of parabolic subgroups. There is a maximal torus action, and hyperkähler quotients by this action give the Kostant varieties, which are the closures in $\mathfrak{sl}(n, \mathbb{C})^*$ of coadjoint orbits of $K_{\mathbb{C}} = SL(n, \mathbb{C})$. We recall that by the work of Kronheimer [17, 18], Biquard [3] and

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Kovalev [14] all coadjoint orbits of complex reductive groups admit hyperkähler structures.

We are led to our construction by algebro-geometric considerations, namely, the wish to produce a variety that is an affine completion of a complex symplectic quotient of the cotangent bundle $T^*K_{\mathbb{C}}$ by the maximal unipotent subgroup N of $K_{\mathbb{C}}$. The upshot is that torus quotients yield not single complex coadjoint orbits but rather their canonical affine completions which are Kostant varieties. In particular, the torus reduction at a triple $(0, \tau_2, \tau_3)$, where $\tau_2 + i\tau_3$ is regular, will give Kronheimer's hyperkähler structure on the coadjoint orbit of $\tau_2 + i\tau_3$ [18]. However torus reduction at the origin yields not a point (as in the symplectic case) but rather the nilpotent variety.

Given a hyperkähler manifold M with a hyperkähler Hamiltonian $SU(n)$ action, we can construct its hyperkähler implosion as the hyperkähler quotient $(M \times Q) /\!/\!/ SU(n)$, by analogy with symplectic implosion. However the connection between non-abelian and abelian quotients is more involved than in the symplectic case. While the torus quotient of the hyperkähler implosion at a triple $(0, \tau_2, \tau_3)$ with $\tau_2 + i\tau_3$ regular coincides with Kronheimer's definition of the hyperkähler reduction of M by $SU(n)$ at this level [18], reducing at level zero will give the hyperkähler quotient of the product of M and the nilpotent variety. To recover the usual hyperkähler quotient we must take just the closed stratum in this space, corresponding to the semisimple stratum (i.e. the point zero) in the nilpotent variety.

We now describe the plan of the paper. In §1 we briefly review the theory of symplectic implosion. In §2 we recall some relevant points from hyperkähler geometry, and introduce various complex-symplectic spaces which will arise as ingredients for building the hyperkähler implosion. In §3 we recall the theory of Kostant varieties and the Grothendieck-Springer resolution. Section 4 shows that we may use symplectic quivers associated to actions of products of special linear groups to give a new model of the symplectic implosion when $K = SU(n)$. In §5, motivated by this construction, we consider hyperkähler quiver varieties and in §6 we stratify them in terms of quiver diagrams. This gives an approach to hyperkähler implosion in the case of $SU(n)$. We analyse the structure of these strata in terms of parabolic subgroups, and identify the implosion with a non-reductive quotient, in §7. Finally in §8 we work out various examples and show how Kostant varieties arise as torus quotients.

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1. Symplectic implosion

We first review the theory of symplectic implosion, due to Guillemin, Jeffrey and Sjamaar [9]. Given a symplectic manifold M with a Hamiltonian symplectic action of a compact Lie group K with maximal torus T, the imploded space M_{impl} is a stratified symplectic space with a Hamiltonian action of the maximal torus T of K. For convenience we fix an invariant inner product on the Lie algebra $\mathfrak{k}$ of K, which allows us to identify $\mathfrak{k}$ with its dual $\mathfrak{k}^*$, and we fix a positive Weyl chamber in the Cartan algebra $\mathfrak{t}$ of T. Then we have an identification of reduced spaces

$$M /\!\!/^s_\lambda K = M_{\mathrm{impl}} /\!\!/^s_\lambda T \tag{1.1}$$

for all λ in the closure of the fixed positive Weyl chamber in $\mathfrak{t}^*$, where $/\!\!/^s_\lambda$ denotes symplectic reduction at level λ. Note that λ need not be central for K; we recall that for general λ the symplectic reduction $M /\!\!/^s_\lambda K$ is the space $(M \times \mathsf{O}_{-\lambda}) /\!\!/^s_0 K$, where O_λ is the coadjoint orbit of K through λ with its canonical symplectic structure. This reduction may be identified with $\mu^{-1}(\lambda)/\mathrm{Stab}_K(\lambda)$ where $\mu\colon M \to \mathfrak{k}^*$ is the moment map for the K-action on M and $\mathrm{Stab}_K(\lambda)$ is the stabiliser in K of $\lambda \in \mathfrak{k}^*$ under the coadjoint action of K.

A particularly important example of implosion is when we take M to be the cotangent bundle T^*K (which may be identified with $K_{\mathbb{C}}$). Now $T^*K /\!\!/^s_\lambda K$ is just the coadjoint orbit O_λ so the imploded space must satisfy

$$(T^*K)_{\mathrm{impl}} /\!\!/^s_\lambda T = \mathsf{O}_\lambda$$

for λ in the closed positive Weyl chamber $\mathfrak{t}^*_+$. Explicitly, $(T^*K)_{\mathrm{impl}}$ is obtained from $K \times \mathfrak{t}^*_+$, by identifying (k_1, ξ) with (k_2, ξ) if k_1, k_2 are related by the translation action of an element of the commutator subgroup of $\mathrm{Stab}_K(\xi)$. So if ξ is in the interior of the chamber, its stabiliser is a torus and we do not perform any identifications. In particular an open dense subset of $(T^*K)_{\mathrm{impl}}$ is just the product of K with the interior of the Weyl chamber.

In fact this example gives us a universal imploded space. As T^*K has a Hamiltonian $K \times K$-action its implosion inherits a Hamiltonian $K \times T$-action. Now for a general symplectic manifold M with a Hamiltonian K-action we obtain the imploded space M_{impl} as the symplectic reduction $(M \times (T^*K)_{\mathrm{impl}}) /\!\!/^s_0 K$, which has an induced Hamiltonian T-action as required.

Implosion also has an interpretation as an algebro-geometric quotient. More precisely, $(T^*K)_{\mathrm{impl}}$ can be identified with an affine variety which

is the quotient

$$K_{\mathbb{C}} /\!\!/ N = \mathrm{Spec}(\mathcal{O}(K_{\mathbb{C}})^N),$$

in the sense of geometric invariant theory (GIT), of the complex reductive group $K_{\mathbb{C}}$ (which is the complexification of K) by its maximal unipotent subgroup N [7, 19]. This variety has a stratification by quotients of $K_{\mathbb{C}}$ by commutators of parabolic subgroups; the open stratum is just $K_{\mathbb{C}}/N$ and $K_{\mathbb{C}} /\!\!/ N$ is the canonical affine completion of the quasi-affine variety $K_{\mathbb{C}}/N$.

2. Towards hyperkähler implosion

We now discuss some issues in constructing a hyperkähler analogue of the symplectic implosion.

2.1. In the symplectic case the key example for implosion was the cotangent bundle T^*K of a compact Lie group. It was shown by Kronheimer [16] that the cotangent bundle $T^*K_{\mathbb{C}}$ of the complexification of K admits a hyperkähler structure. (For further aspects of the geometry of this space, especially the moment geometry, see [5]). In fact Kronheimer showed that $T^*K_{\mathbb{C}}$ may be identified with the moduli space $\mathcal{M}_{(K)}$ of solutions to the Nahm equations

$$\frac{dT_i}{dt} + [T_0, T_i] = [T_j, T_k], \qquad (ijk) \text{ cyclic permutation of } (123),$$

(that is, the anti-self-dual-Yang-Mills equations with $\mathbb{R}^3$ translation invariance imposed) where T_i (for $i = 0, 1, 2, 3$) takes values in $\mathfrak{k}$ and is smooth on the interval $[0, 1]$. Two solutions are identified if they are equivalent under the gauge action

$$T_0 \mapsto gT_0g^{-1} - \dot{g}g^{-1}, \qquad T_i \mapsto gT_ig^{-1} \ (i = 1, 2, 3),$$

where $g\colon [0, 1] \mapsto K$ with $g(0) = g(1) = 1 \in K$. The Nahm equations may be viewed as the vanishing condition for a hyperkähler moment map for the action of this group of gauge transformations on an infinite-dimensional flat quaternionic space of $\mathfrak{k}$-valued functions on $[0, 1]$. In this way $\mathcal{M}_{(K)}$ acquires a hyperkähler structure. The complex-symplectic structure defined by the hyperkähler structure on $\mathcal{M}_{(K)}$ is just the standard complex-symplectic form on $T^*K_{\mathbb{C}}$.

We have an action of $K \times K$ on $\mathcal{M}_{(K)} \cong T^*K_{\mathbb{C}}$, defined by taking gauge transformations which are no longer constrained to be the identity at $t = 0, 1$. This action preserves the hyperkähler structure, and complexifies to a holomorphic action of $K_{\mathbb{C}} \times K_{\mathbb{C}}$. Viewing $T^*K_{\mathbb{C}}$ as $K_{\mathbb{C}} \times \mathfrak{k}^*_{\mathbb{C}}$, the left and right actions are given by

$$(g, \xi) \mapsto (h_L g h_R^{-1}, Ad(h_R)^* \xi). \tag{2.1}$$

There is also an action of $SO(3)$), given for $(a_{ij}) \in SO(3)$ by

$$T_i \mapsto \sum_{j=1}^{3} a_{ij} T_j,$$

which is isometric but rotates the two-sphere of complex structures associated to the hyperkähler structure, so that all the complex structures are equivalent. In terms of the above procedure for identifying the moduli space with $T^*K_{\mathbb{C}}$, the choice of complex structures corresponds to the choice in the splitting of the Nahm equations into a real and a complex equation. For the standard choice of complex structures $(\mathsf{I}, \mathsf{J}, \mathsf{K})$ with corresponding symplectic forms $(\omega_{\mathsf{I}}, \omega_{\mathsf{J}}, \omega_{\mathsf{K}})$ the complex-symplectic form is $\omega_{\mathsf{J}} + \mathbf{i}\omega_{\mathsf{K}}$.

2.2. We have already recalled that the symplectic implosion of $T^*K \cong K_{\mathbb{C}}$ can be identified with the non-reductive GIT quotient $K_{\mathbb{C}} /\!\!/ N$ and has a stratification into strata $K_{\mathbb{C}}/[P,P]$ where P ranges over the standard parabolic subgroups of $K_{\mathbb{C}}$. The open stratum, for which P is the Borel subgroup B of $K_{\mathbb{C}}$ associated to the choice of positive Weyl chamber $\mathfrak{t}_+$, is $K_{\mathbb{C}}/N$ where $N = [B,B]$ is a maximal unipotent subgroup of $K_{\mathbb{C}}$.

In the hyperkähler setting, it is natural therefore to consider complex-symplectic quotients of $T^*K_{\mathbb{C}}$ by N and more generally by commutators of parabolic subgroups $P \supseteq B$, acting on the right.

The complex-symplectic moment map for the right action of $K_{\mathbb{C}}$ on $T^*K_{\mathbb{C}}$ is just the I-holomorphic moment map

$$(g, \xi) \mapsto \xi,$$

which of course is equivariant for the right action and invariant for the left action as described in (2.1). So the complex-symplectic quotient by the maximal unipotent group N at level 0 is $K_{\mathbb{C}} \times_N \mathfrak{n}^\circ$, where the annihilator $\mathfrak{n}^\circ$ in $\mathfrak{k}_{\mathbb{C}}^*$ of the Lie algebra $\mathfrak{n}$ of N may be identified with the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{k}_{\mathbb{C}}$. Here we use a fixed invariant inner product on $\mathfrak{k}$ to identify $\mathfrak{k}$ with $\mathfrak{k}^*$ and to identify $\mathfrak{k}_{\mathbb{C}}$ with $\mathfrak{k}_{\mathbb{C}}^*$; when $K = SU(n)$ this identification is given by the pairing $(A,B) \mapsto \operatorname{tr}(AB)$ on the Lie algebra of $SL(n,\mathbb{C})$.

The complex-symplectic form $\omega_{\mathsf{J}} + \mathbf{i}\omega_{\mathsf{K}}$ on $T^*K_{\mathbb{C}}$ descends to the complex-symplectic quotient $K_{\mathbb{C}} \times_N \mathfrak{n}^\circ$. We can also perform complex-symplectic quotients by commutators $[P,P]$ of general parabolics P. We obtain quotients $K_{\mathbb{C}} \times_{[P,P]} [\mathfrak{p},\mathfrak{p}]^\circ$, which may be identified with the cotangent bundles $T^*(K_{\mathbb{C}}/[P,P])$ of the strata of the symplectic implosion.

Of course, these quotients carry a complex-symplectic action of $K_{\mathbb{C}}$ induced from the left action on $T^*K_{\mathbb{C}}$. As the maximal torus $T_{\mathbb{C}}$ normalises N (and $[P,P]$) we also have a surviving right action of $T_{\mathbb{C}}$. So these quotients have a complex-symplectic action of $K_{\mathbb{C}} \times T_{\mathbb{C}}$.

2.3. On the other hand the symplectic implosion is given by the non-reductive GIT quotient $K_{\mathbb{C}} /\!\!/ N$ which contains $K_{\mathbb{C}}/N$ as an open subset. So when searching for a candidate for the universal hyperkähler implosion we might look for a quotient in the sense of GIT of $K_{\mathbb{C}} \times \mathfrak{n}^\circ$ by the action of N. However classical GIT [19] only deals with actions of reductive groups, and the unipotent group N is not reductive. There is no difficulty in constructing a non-reductive GIT quotient $K_{\mathbb{C}} /\!\!/ N$ of $K_{\mathbb{C}}$ by N, since the algebra $\mathcal{O}(K_{\mathbb{C}})^N$ of N-invariant regular functions on $K_{\mathbb{C}}$ is finitely generated and so we can define $K_{\mathbb{C}} /\!\!/ N$ to be the associated affine variety

$$K_{\mathbb{C}} /\!\!/ N = \operatorname{Spec}(\mathcal{O}(K_{\mathbb{C}})^N).$$

This means that if X is any complex affine variety on which $K_{\mathbb{C}}$ acts then the algebra of invariants

$$\mathcal{O}(X)^N \cong (\mathcal{O}(X) \otimes \mathcal{O}(K_{\mathbb{C}})^N)^{K_{\mathbb{C}}}$$

is finitely generated and we have a non-reductive GIT quotient

$$X /\!\!/ N = \operatorname{Spec}(\mathcal{O}(X)^N) \cong (X \times (K_{\mathbb{C}} /\!\!/ N)) /\!\!/ K_{\mathbb{C}}.$$

Unfortunately the N action

$$(g, \xi) \mapsto (gn^{-1}, n\xi n^{-1})$$

on $K_{\mathbb{C}} \times \mathfrak{n}^\circ$ does not extend to a $K_{\mathbb{C}}$ action, so constructing a non-reductive quotient $(K_{\mathbb{C}} \times \mathfrak{n}^\circ) /\!\!/ N$ is not so straightforward (although see [7]). However we will prove in this paper that when $K = SU(n)$ the algebra $\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^\circ)^N$ is finitely generated, and that

$$(K_{\mathbb{C}} \times \mathfrak{n}^\circ) /\!\!/ N = \operatorname{Spec}(\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^\circ)^N)$$

can be identified with a hyperkähler quotient of a flat space $\mathbb{H}^m$ by a compact group action using quiver diagrams. This hyperkähler quotient is a stratified hyperkähler space with a hyperkähler torus action, and is a complex affine variety for any choice of complex structure. It also includes the quotients $K_{\mathbb{C}} \times_{[P,P]} [\mathfrak{p}, \mathfrak{p}]^\circ$ discussed above, in particular $K_{\mathbb{C}} \times_N \mathfrak{n}^\circ$. It is also the canonical affine completion of $K_{\mathbb{C}} \times_N \mathfrak{n}^\circ$.

3. Kostant varieties

Let us now discuss some links with geometric representation theory (cf. [4] for background).

We have already observed that we expect the universal hyperkähler implosion to be a non-reductive GIT quotient

$$(K_{\mathbb{C}} \times \mathfrak{n}^\circ) /\!\!/ N = \operatorname{Spec}(\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^\circ)^N)$$

with the actions of the maximal torus T of K and its complexification $T_{\mathbb{C}} = B/N$ induced from the action of the Borel subgroup B on $K_{\mathbb{C}} \times \mathfrak{n}^\circ$. We also will be concerned with the geometric quotient $K_{\mathbb{C}} \times_N \mathfrak{n}^\circ$. In this

section we shall study quotients and complex-symplectic reductions of these spaces by tori.

First consider the space $\tilde{\mathfrak{k}}_{\mathbb{C}} = K_{\mathbb{C}} \times_B \mathfrak{n}^\circ = K_{\mathbb{C}} \times_B \mathfrak{b}$, where B is the Borel subgroup with Lie algebra $\mathfrak{b}$. Now $\tilde{\mathfrak{k}}_{\mathbb{C}}$ may be identified via $(Q, X) \mapsto (QXQ^{-1}, QB/B)$ with the correspondence space

$$\{(X, \mathfrak{b}) \in \mathfrak{k}_{\mathbb{C}} \times \mathcal{B} : X \in \mathfrak{b}\},$$

where $\mathcal{B} = K_{\mathbb{C}}/B$ is the variety of Borel subalgebras in $\mathfrak{k}_{\mathbb{C}}$. Projection onto the second factor realises $\tilde{\mathfrak{k}}_{\mathbb{C}}$ as a vector bundle over $\mathcal{B}$. Projection onto the first factor, on the other hand, gives a map

$$\mu\colon \tilde{\mathfrak{k}}_{\mathbb{C}} = K_{\mathbb{C}} \times_B \mathfrak{b} \to \mathfrak{k}_{\mathbb{C}},$$
$$\mu\colon (Q, X) \mapsto QXQ^{-1},$$

called the Grothendieck simultaneous resolution. This map is a closed and proper surjection (since $\mathcal{B}$ is compact). Over regular elements of $\mathfrak{k}_{\mathbb{C}}$ it is finite-to-one, of degree $|W|$, where W is the Weyl group.

We also have a map

$$\rho\colon\ \mathfrak{k}_{\mathbb{C}} \to \mathbb{C}^r \simeq \mathfrak{t}_{\mathbb{C}}/W,$$

where $r = \operatorname{rank} \mathfrak{k}_{\mathbb{C}}$. This map is defined by choosing generators $p_1, \ldots, p_r$ for the ring of invariant polynomials on $\mathfrak{k}_{\mathbb{C}}$ and setting

$$\rho(X) = (p_1(X), \ldots, p_r(X)).$$

Now let us fix $X_0 \in \mathfrak{t}_{\mathbb{C}}$ and consider the subset of $\tilde{\mathfrak{k}}_{\mathbb{C}}$ given by

$$\tilde{\mathfrak{k}}_{\mathbb{C}}(X_0) = K_{\mathbb{C}} \times_B (X_0 + \mathfrak{n}).$$

This has the structure of an affine bundle over $\mathcal{B} = K_{\mathbb{C}}/B$ (when $X_0 = 0$ it is the cotangent bundle $T^*\mathcal{B}$).

Lemma 3.1. *We have a surjection*

$$\mu\colon \bigcup_{w \in W} \tilde{\mathfrak{k}}_{\mathbb{C}}(w.X_0) \to \rho^{-1}(\chi),$$

where $\chi = \rho(X_0)$.

Proof. This follows from the commutative diagram [4, (3.1.41)]:

$$\begin{tikzcd} & \tilde{\mathfrak{k}}_{\mathbb{C}} \arrow[dl, "\mu"] \arrow[dr, "\nu"] & \\ \mathfrak{k}_{\mathbb{C}} \arrow[dr, "\rho"] & & \mathfrak{t}_{\mathbb{C}} \arrow[dl, "\pi"] \\ & \mathfrak{t}_{\mathbb{C}}/W = \mathbb{C}^r & \end{tikzcd}$$

Here ν is the map that sends X to its component X_0 in the Cartan algebra, and π is just the quotient by the Weyl action.

Explicitly, we can argue as follows. If p_i is an invariant polynomial as above, then, letting $X = X_0 + Y$ where $Y \in \mathfrak{n}$, we have

$$p_i \circ \mu(Q, X) = p_i(QXQ^{-1}) = p_i(X) = p_i(X_0)$$

where the last equality comes from [4, Corollary 3.1.43]. So $\mu(\tilde{\mathfrak{k}}_{\mathbb{C}}(X_0))$ is contained in the fibre $\rho^{-1}(\chi)$ of ρ, where $\chi = \rho(X_0)$. This argument shows $\mu(\tilde{\mathfrak{k}}_{\mathbb{C}}(w.X_0))$ is also contained in $\rho^{-1}(\chi)$, where w is an element of the Weyl group W.

Conversely, if $\rho(X) = \chi$, write $X = \mu(Q, X') = QX'Q^{-1}$ for some $(Q, X') \in K_{\mathbb{C}} \times \mathfrak{b}$. Write $X' = X_{ss} + Y$, where $X_{ss} \in \mathfrak{t}_{\mathbb{C}}$ and $Y \in \mathfrak{n}$. As above $\rho(X_{ss}) = \rho(X') = \rho(X) = \chi = \rho(X_0)$, so $X_{ss}, X_0 \in \mathfrak{t}_{\mathbb{C}}$ are equivalent under the Weyl group action. $\square$

We recall some facts, due to Kostant [13], concerning the *Kostant varieties* $V_\chi = \rho^{-1}(\chi)$ (see [4, §6.7]):

(i) V_χ is an irreducible normal affine variety of complex dimension $\dim_{\mathbb{C}} \mathfrak{k}_{\mathbb{C}} - r = \dim_{\mathbb{C}} \mathfrak{k}_{\mathbb{C}} - \dim \mathfrak{t}_{\mathbb{C}}$,

(ii) V_χ is a union of finitely many orbits for the adjoint action of $K_{\mathbb{C}}$,

(iii) there is a unique open dense orbit V_χ^{reg}, and this consists of the regular elements in V_χ,

(iv) the complement of the open dense orbit V_χ^{reg} is of complex codimension $\geqslant 2$ in V_χ [13, Theorem 0.8], and hence $\mathcal{O}(V_\chi^{\mathrm{reg}})$ is finitely generated and V_χ is the canonical affine completion $\operatorname{Spec}(\mathcal{O}(V_\chi^{\mathrm{reg}}))$ of the quasi-affine variety V_χ^{reg},

(v) there is a unique closed orbit; this consists of the semisimple elements in V_χ and has minimal dimension among the orbits in V_χ,

(vi) V_χ consists of a single orbit if and only if it contains a regular semisimple element.

Note also that $\dim_{\mathbb{C}} \tilde{\mathfrak{k}}_{\mathbb{C}}(X_0) = 2 \dim_{\mathbb{C}} \mathfrak{n} = \dim_{\mathbb{C}} \mathfrak{k}_{\mathbb{C}} - r = \dim V_\chi$.

We know μ maps $\tilde{\mathfrak{k}}_{\mathbb{C}}(X_0)$ into V_χ. But the above discussion of Lemma 3.1 shows its image is not contained in a proper subvariety of V_χ. Hence the image contains a Zariski-open and hence dense set in V_χ. But μ is closed on $\tilde{\mathfrak{k}}_{\mathbb{C}}$ and hence on the closed subset $\tilde{\mathfrak{k}}_{\mathbb{C}}(X_0)$, so the image is all of V_χ. This map is in fact injective over regular elements (the fibre for $\mu\colon \tilde{\mathfrak{k}}_{\mathbb{C}} \to \mathfrak{k}_{\mathbb{C}}$ over regulars is finite-to-one with the fibres coming from the Weyl group). So we have

Lemma 3.2. *We have a surjection*

$$\mu\colon \tilde{\mathfrak{k}}_{\mathbb{C}}(X_0) \to V_\chi$$

onto the Kostant variety, which is injective over regular elements.

By analogy with symplectic implosion we expect that hyperkähler quotients of the universal hyperkähler implosion by the maximal torus T in K should be closely related to coadjoint orbits of the complexified

group $K_{\mathbb{C}}$. Reduction at $(0, \zeta_2, \zeta_3)$ should correspond to the complex-symplectic quotient by $T_{\mathbb{C}}$ at level $X_0 = \zeta_2 + \mathbf{i}\zeta_3$. So let us consider the quotients $\tilde{\mathfrak{k}}_{\mathbb{C}}(X_0) = K_{\mathbb{C}} \times_B (X_0 + \mathfrak{n})$ and also the GIT quotient

$$(K_{\mathbb{C}} \times (X_0 + \mathfrak{n}))/\!\!/B. \tag{3.3}$$

From above $\hat{\mu}\colon (Q, X) \mapsto QXQ^{-1}$ gives a B-invariant map from $K_{\mathbb{C}} \times (X_0 + \mathfrak{n})$ onto V_χ where $\chi = \rho(X_0)$. This descends to the map $\mu\colon \tilde{\mathfrak{k}}_{\mathbb{C}}(X_0) = K_{\mathbb{C}} \times_B (X_0 + \mathfrak{n}) \to V_\chi$.

If X_0 is regular semisimple, then, from (v) above, V_χ is just the orbit through X_0. Moreover in this case we have an isomorphism

$$\mu\colon K_{\mathbb{C}} \times_B (X_0 + \mathfrak{n}) \to V_\chi,$$

hence the fibres of $\hat{\mu}$ are exactly the B-orbits. So (3.3) is a geometric quotient and our hyperkähler quotient is just $V_\chi = K_{\mathbb{C}}/T_{\mathbb{C}}$.

If X_0 is not regular semisimple, then V_χ will contain more than one orbit. As mentioned above in (iii), the regular elements will form an open dense set, but the smaller strata in its closure will consist of non-regular elements.

Now $\hat{\mu}\colon (Q, X) \mapsto QXQ^{-1}$ still defines a B-invariant map from $K_{\mathbb{C}} \times (X_0 + \mathfrak{n})$ onto V_χ, inducing $\mu\colon \tilde{\mathfrak{k}}_{\mathbb{C}}(X_0) \to V_\chi$. This time μ is no longer an isomorphism, but it is injective over the set V_χ^{reg} of regular elements, hence injective over the complement of a set of codimension $\geqslant 2$ in V_χ.

Clearly a polynomial f on V_χ induces a B-invariant polynomial $\tilde{f}$ on $K_{\mathbb{C}} \times (X_0 + \mathfrak{n})$ via $\tilde{f}(Q, X) = f(QXQ^{-1})$. Conversely, a B-invariant polynomial on $K_{\mathbb{C}} \times (X_0 + \mathfrak{n})$ will induce a polynomial on $\tilde{\mathfrak{k}}_{\mathbb{C}}(X_0) = K_{\mathbb{C}} \times_B (X_0 + \mathfrak{n})$ and hence a polynomial on V_χ^{reg}. This will extend over the codimension $\geqslant 2$ locus of non-regular points to give a polynomial on V_χ

Hence V_χ is the GIT quotient $(K_{\mathbb{C}} \times (X_0 + \mathfrak{n}))/\!\!/B$. Thus if, as we expect, the universal hyperkähler implosion can be identified with a non-reductive GIT quotient $(K_{\mathbb{C}} \times \mathfrak{b})/\!\!/N$, then its complex-symplectic GIT reduction by the torus $T_{\mathbb{C}}$ at level X_0 is V_χ.

The space $\tilde{\mathfrak{k}}_{\mathbb{C}}(X_0) = K_{\mathbb{C}} \times_B (X_0 + \mathfrak{n})$, on the other hand, is an affine bundle over $\mathcal{B}$ and is a desingularisation of the Kostant variety V_χ.

Example 3.4. In the particular case $X_0 = 0$, then $\tilde{\mathfrak{k}}_{\mathbb{C}}(X_0)$ is just $K_{\mathbb{C}} \times_B \mathfrak{n}$, which is the cotangent bundle $T^*\mathcal{B} = T^*(K_{\mathbb{C}}/B)$. Now the restriction of μ to $\tilde{\mathfrak{k}}_{\mathbb{C}}(0)$ is the Springer resolution

$$\mu\colon T^*\mathcal{B} \to \mathcal{N},$$

where $\mathcal{N} = V_0 = \rho^{-1}(0)$ is the nilpotent variety in $\mathfrak{k}_{\mathbb{C}}$. Both these spaces appear as hyperkähler spaces in the work of Nakajima [20].

If $K = SU(2)$ then $T^*\mathcal{B}$ is $T^*\mathbb{P}^1$, the resolution of the nilpotent cone which is the GIT quotient at level zero (see Example 8.5 below). ♢

Remark 3.5. If our candidate for the universal hyperkähler implosion were the geometric quotient $K_{\mathbb{C}} \times_N \mathfrak{n}^\circ = K_{\mathbb{C}} \times_N \mathfrak{b}$ instead of the non-reductive GIT quotient $(K_{\mathbb{C}} \times \mathfrak{b})/\!\!/ N$, then a naive complex-symplectic reduction at level X_0 (taking a geometric rather than GIT quotient) would give the Springer resolution $K_{\mathbb{C}} \times_B (X_0 + \mathfrak{n}) = \tilde{\mathfrak{k}}_{\mathbb{C}}(X_0)$ rather than the Kostant variety V_χ.

4. Symplectic quivers

In this section we shall present a new model for the universal symplectic implosion for $K = SU(n)$, in terms of *symplectic quiver representations*. This will also introduce some ideas which will be useful in the next section, when we introduce a quiver description of the universal hyperkähler implosion for $SU(n)$.

A *symplectic quiver representation* is a diagram of vector spaces and linear maps

$$0 = V_0 \stackrel{\alpha_0}{\to} V_1 \stackrel{\alpha_1}{\to} V_2 \stackrel{\alpha_2}{\to} \cdots \stackrel{\alpha_{r-2}}{\to} V_{r-1} \stackrel{\alpha_{r-1}}{\to} V_r = \mathbb{C}^n. \tag{4.1}$$

We say that the vector spaces V_i have dimension vector $\mathbf{n} = (n_1, \dots, n_r)$ if $n_i = \dim V_i$. Let

$$R(\mathbf{n}) = \bigoplus_{i=1}^{r} \operatorname{Hom}(V_{i-1}, V_i)$$

be the space of all such diagrams with $V_i = \mathbb{C}^{n_i}$ for $1 \leqslant i \leqslant r$. We will say that the representation is *ordered* if $0 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r = n$ and *strictly ordered* if $0 < n_1 < n_2 < \cdots < n_r = n$.

We shall be interested in the GIT quotient of $R(\mathbf{n})$ by an action of

$$SL := \prod_{i=1}^{r-1} SL(V_i).$$

This is a subgroup of $GL := \prod_{i=1}^{r-1} GL(V_i)$ and for both groups $g = (g_1, \dots, g_{r-1})$ acts by

$$\begin{aligned} \alpha_i &\mapsto g_{i+1}\alpha_i g_i^{-1} \quad (i = 1, \dots, r-2), \\ \alpha_{r-1} &\mapsto \alpha_{r-1} g_{r-1}^{-1}. \end{aligned}$$

There is also of course a commuting action of $GL(n, \mathbb{C}) = GL(V_r)$ by left multiplication of α_{r-1}.

We shall particularly consider the case of full flag representations $\mathbf{n} = (1, 2, \dots, n-1, n)$, but their analysis will require the study of the more general quiver representations (4.1) for general ordered $\mathbf{n}$.

The points in the GIT quotient $R(\mathbf{n})/\!\!/ SL = \operatorname{Spec}(\mathcal{O}(R(\mathbf{n})^{SL}))$ correspond to the closed orbits for the SL action on the affine variety $R(\mathbf{n})$. The term *polystable* is often used to describe points of $R(\mathbf{n})$ which lie in closed SL-orbits. If in addition the stabiliser is finite, the point is called *stable*.

We first look at length 2 quivers.

Lemma 4.2. *A length two diagram*

$$V \xrightarrow{\alpha} \mathbb{C}^n$$

gives a closed $SL(V)$ orbit if and only if α is either 0 or injective.

Proof. Write $V = \ker\alpha \oplus U$ for some U.

Then

$$\alpha = \begin{pmatrix} 0 & a_2 \end{pmatrix} : \ker\alpha \oplus U \to \mathbb{C}^n$$

which transforms as

$$\begin{pmatrix} 0 & a_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & a_2 g_2^{-1} \end{pmatrix}$$

under the action of $g = \operatorname{diag}(g_1, g_2) \in SL(V)$. If $\ker\alpha \neq 0$ and $U \neq 0$, then $\operatorname{diag}(\lambda^{-k}, \lambda^{\ell}) \in SL(V)$ where $\ell = \dim\ker\alpha$ and $k = \dim U$, and the corresponding one-parameter group has 0 as its limit. Thus, if the orbit is closed, either $\ker\alpha = 0$ and α is injective or $\alpha = 0$.

If $\alpha = 0$ then the orbit is clearly closed. If α is injective, consider the action of any one-parameter group of $SL(V)$ (in the sense of GIT, i.e. the image of a homomorphism $\mathbb{C}^* \to SL(V)$). Write $V = \bigoplus_{i=1}^r E(\mu_i)$ as a direct sum of weight spaces. As the one-parameter group is in $SL(V)$ we have $\sum_i \mu_i \dim E(\mu_i) = 0$. Now $\alpha = \bigoplus_{i=1}^r a_i$ with each a_i non-zero and under the action of the one-parameter group we have $a_i \mapsto \lambda^{\mu_i} a_i$. This has a limit as $\lambda \to \infty$ only if $\mu_i \leqslant 0$ for each i. But the special linear condition then gives $\mu_i = 0$ for each i. Thus by the Mumford numerical criterion [19] α is stable and hence polystable. $\square$

To deal with the general case, we consider the following length 2 situation for a double quiver. Spaces of such double quivers will also give hyperkähler varieties, and will be studied further in the next section.

Lemma 4.3. *A configuration*

$$V \underset{\beta}{\overset{\alpha}{\rightleftarrows}} W \tag{4.4}$$

of vector spaces V, W and linear maps α, β gives a closed orbit under the $SL(V)$-action

$$\alpha \mapsto \alpha g^{-1} \qquad \beta \mapsto g\beta$$

if and only if

(i) *α is injective, or*
(ii) *β is surjective, or*
(iii) *$V = \ker\alpha \oplus \operatorname{im}\beta$.*

Proof. Suppose (α, β) is SL-polystable. Choose a direct sum decomposition $V = U_1 \oplus U_2 \oplus U_3 \oplus U_4$ such that

$$\ker\alpha \cap \operatorname{im}\beta = U_1, \quad \ker\alpha = U_1 \oplus U_2 \quad \text{and} \quad \operatorname{im}\beta = U_1 \oplus U_3.$$

Then we may write

$$\alpha = \begin{pmatrix} 0 & 0 & a_3 & a_4 \end{pmatrix}, \quad \beta = \begin{pmatrix} b_1 \\ 0 \\ b_3 \\ 0 \end{pmatrix}.$$

A block-diagonal element $\operatorname{diag}(g_1, \dots, g_4) \in SL(V)$ acts as

$$\alpha \mapsto \begin{pmatrix} 0 & 0 & a_3 g_3^{-1} & a_4 g_4^{-1} \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} g_1 b_1 \\ 0 \\ g_3 b_3 \\ 0 \end{pmatrix}.$$

If U_2 is non-zero, then we may choose $g_1 \in GL(U_1)$ and $g_4 \in GL(U_4)$ freely and take $g_2 = \lambda$, $g_3 = 1$, where $\lambda^k \det g_1 \det g_4 = 1$, and $k = \dim U_2$. In particular, we may choose a one-parameter group of this form such that $g_1 \to 0$ and $g_4^{-1} \to 0$. Closure of the orbit implies that $b_1 = 0$ and $a_4 = 0$. But b_1 is onto and a_4 is injective, so $U_1 = 0 = U_4$ and $V = \ker \alpha \oplus \operatorname{im} \beta$.

If $U_2 = 0$ and U_1, U_4 are both non-zero, then we may consider $g = \operatorname{diag}(g_1, \dots, g_4) \in SL(V)$ with $g_3 = 1$. Now if $g_1 \to 0$, we have $g_4^{-1} \to 0$ too and closure of the orbit implies $b_1 = 0 = a_4$, contradicting the assumption that U_1, U_4 are non-zero.

So we see that $U_2 = 0$ implies either $U_1 = U_2 = 0$, giving α injective, or $U_2 = U_4 = 0$, giving β surjective.

We note that the proof of Lemma 4.2 shows that the case when α is injective gives a stable configuration. For the case when β is surjective, note that stability of (4.4) is equivalent to stability of the dual diagram

$$V^* \underset{\alpha^*}{\overset{\beta^*}{\rightleftarrows}} W^*$$

and that β surjective is equivalent to β^* injective.

Finally, for stability of the mixed case when $V = \ker \alpha \oplus \operatorname{im} \beta$, we may assume $\ker \alpha$ is non-trivial and β is not surjective. Consider the endomorphism $X = \alpha\beta$ of W. This is invariant under the action of $SL(V)$ and we have $\operatorname{rank} X = \operatorname{rank} \alpha = \operatorname{rank} \beta$.

Suppose $g_t \in SL(V)$ is a one-parameter group and that $g_t(\alpha, \beta) \to (\alpha', \beta')$. We have $\operatorname{rank} \alpha' \leqslant \operatorname{rank} \alpha$ and $\operatorname{rank} \beta' \leqslant \operatorname{rank} \beta$. But $\alpha'\beta' = X$, so $\operatorname{rank} X \leqslant \min\{\operatorname{rank} \alpha', \operatorname{rank} \beta'\}$ giving $\operatorname{rank} \alpha' = \operatorname{rank} \alpha$ and $\operatorname{rank} \beta' = \operatorname{rank} \beta$. Also observe that $\ker \beta' = \ker \beta = \ker X$, by our direct sum decomposition. It now follows that $\beta' = g\beta$ for some $g \in GL(V)$. Since β is not surjective, we may in fact choose $g \in SL(V)$. Now $X = \alpha\beta = \alpha'\beta' = \alpha' g \beta$ implies $\alpha = \alpha' g$ on $\operatorname{im} \beta$. But α and α' have the same rank as β, and α is injective on $\operatorname{im} \beta$, so $\ker(\alpha' g)$ is a complementary subspace to $\operatorname{im} \beta$. Choosing $h \in SL(V)$ extending the identity on $\operatorname{im} \beta$ and with $h \ker \alpha = \ker(\alpha' g)$, we get that $\alpha' = \alpha(gh)^{-1}$, $\beta' = gh\beta$ and so (α', β') lies in the $SL(V)$-orbit of (α, β). $\square$

Theorem 4.5. *The symplectic quiver representation $\alpha \in R(\mathbf{n})$ (4.1) is SL-polystable only if at each stage $i = 1, \dots, r-1$ we have either*

(i) *α_i is injective, or*
(ii) *$V_i = \operatorname{im} \alpha_{i-1} \oplus \ker \alpha_i$, or*
(iii) *α_{i-1} is surjective.*

Note that if $\mathbf{n}$ is strictly ordered, then the final possibility cannot occur.

Proof. The necessity follows from Lemma 4.3 applied to $V = V_i$, $W = V_{i+1} \oplus V_{i-1}$ and the maps $\alpha = (\alpha_i, 0)$, $\alpha(x) \coloneqq (\alpha_i(x), 0)$ and $\beta = (0, \alpha_{i-1})$, $\beta(x, y) \coloneqq \alpha_{i-1}(y)$. $\square$

We shall now consider full flag quivers; that is, those where $V_i = \mathbb{C}^i$ for $i = 1, \dots, n$. To describe the GIT quotient by $SL = \prod_{i=2}^{n-1} SL(i, \mathbb{C})$ we need to analyse the quotient by SL of the set of such quivers satisfying the conditions given by Theorem 4.5; in the course of the analysis we shall see that the condition of Theorem 4.5 is sufficient as well as necessary for polystability.

First, observe that we may decompose each vector space $\mathbb{C}^i$ as

$$\mathbb{C}^i = \ker \alpha_i \oplus \mathbb{C}^{m_i}, \tag{4.6}$$

where $\mathbb{C}^{m_i} = \mathbb{C}^i$ if α_i is injective and we take $\mathbb{C}^{m_i} = \operatorname{im} \alpha_{i-1}$ otherwise. We put $m_n = n$. Note that rank $\alpha_i = m_i$ for $1 \leqslant i \leqslant n-1$. Further, observe that this actually gives a decomposition of our quiver into two subquivers, namely

$$\mathbb{C}^{m_1} \xrightarrow{\bar\alpha_1} \mathbb{C}^{m_2} \xrightarrow{\bar\alpha_2} \cdots \xrightarrow{\bar\alpha_{n-2}} \mathbb{C}^{m_{n-1}} \xrightarrow{\bar\alpha_{n-1}} \mathbb{C}^n, \tag{4.7}$$

where $\bar\alpha_i$ denotes the restriction of α_i to $\mathbb{C}^{m_i}$, and the quiver with trivial maps

$$\ker \alpha_1 \xrightarrow{0} \ker \alpha_2 \xrightarrow{0} \cdots \xrightarrow{0} \ker \alpha_{n-1} \xrightarrow{0} 0.$$

Note that we may use the SL action to standardise the decomposition (4.6), and we have a residual action of $\prod_{i=1}^{n-1} S(GL(m_i, \mathbb{C}) \times GL(i - m_i, \mathbb{C}))$ preserving the decomposition. The $GL(i - m_i, \mathbb{C})$ action on the quiver with zero maps is trivial.

If α_i is not injective, then, since $\mathbb{C}^i = \ker \alpha_i \oplus \operatorname{im} \alpha_{i-1}$ we see that $\operatorname{rank} \alpha_i = \operatorname{rank} \alpha_{i-1}$. We deduce that

$$m_i = i, \qquad \text{if } \alpha_i \text{ is injective,} \tag{4.8}$$

$$m_i = m_{i-1}, \qquad \text{if } \alpha_i \text{ is not injective.} \tag{4.9}$$

Now all the information is contained in the quiver (4.7) with $SL(m_i, \mathbb{C})$ acting for $m_i = i$ and $GL(m_i, \mathbb{C})$ acting for $m_i < i$. All the restricted maps $\bar\alpha_i$ are injective, so $m_{i-1} \leqslant m_i$ for each i. When $m_i = m_{i-1}$, this gives $m_i \leqslant i - 1 < i$, so we have a $GL(m_i, \mathbb{C})$ action on $\mathbb{C}^{m_i}$. We may use up this action by standardising $\bar\alpha_i$ to be the identity. We may therefore remove this edge of the quiver, a process we call *contraction*.

After performing all such contractions, we arrive at a length r quiver where $m_1 < m_2 < \cdots < m_{r-1} < m_r = n$ and all maps $\bar{\alpha}_i$ are injective. The residual action is $\prod_{i=1}^{r-1} SL(m_i, \mathbb{C})$. For each j, we have $\bar{\alpha}_{r-1}\bar{\alpha}_{r-2}\dots\bar{\alpha}_j g_j^{-1} = (\bar{\alpha}_{r-1}g_{r-1}^{-1})\prod_{i=r-2}^{j} g_{i+1}\bar{\alpha}_i g_i^{-1}$, so if this quiver tends to a limit under the action of a one-parameter subgroup $(g_i(t))_{i=1}^{r-1}$, then the injective map $\bar{\alpha}_{r-1}\bar{\alpha}_{r-2}\dots\bar{\alpha}_j g_j^{-1}(t)$ must tend to a limit also. Now the argument of the last section of Lemma 4.2 shows that $g_j(t)$ is trivial. So there are no destabilising one-parameter subgroups and we have that the quiver is polystable.

Alternatively, we may use the action of $SL(n, \mathbb{C}) \times \prod_{i=1}^{r-1} SL(m_i, \mathbb{C})$ to put the maps $\bar{\alpha}_i$ into a form where the only non-zero entries are the (j, j) terms for $j = 1, \dots, m_i$. Moreover the (j, j) terms may be chosen to be arbitrary non-zero scalars. It is now straightforward to check that these scalars may be chosen so that the real moment map for the action of $\prod_{i=1}^{r-1} SU(m_i)$ vanishes, i.e. so that the trace-free part of $\alpha_{i-1}\alpha_{i-1}^* - \alpha_i^*\alpha_i$ is zero for $i = 1, \dots, r-1$. Thus there exists some $h \in SL(n, \mathbb{C})$ such that the $\prod_{i=1}^{r-1} SL(m_i, \mathbb{C})$ orbit through the image of the quiver under the action of h is closed, and it follows that the orbit through the original quiver is closed too. So the condition of Theorem 4.5 is sufficient for polystability.

Our discussion now shows that we have a stratification of the GIT quotient by $\prod_{i=2}^{n-1} SL(i, \mathbb{C})$ of the space of full flag quivers. There are 2^{n-1} strata, corresponding to the strictly increasing sequences of positive integers ending with n, or equivalently to the ordered partitions of n. We may also of course view the strata as being indexed by the standard parabolic subgroups of $SL(n, \mathbb{C})$.

We next analyse these strata.

Lemma 4.10. *If* $0 < n_1 < n_2 < \cdots < n_r = n$ *then the space of quivers*

$$0 \to V_1 \xrightarrow{\bar{\alpha}_1} V_2 \xrightarrow{\bar{\alpha}_2} \cdots \xrightarrow{\bar{\alpha}_{r-2}} V_{r-1} \xrightarrow{\bar{\alpha}_{r-1}} V_r = \mathbb{C}^n$$

with $\dim V_j = n_j$ *and all* $\bar{\alpha}_i$ *injective, modulo the action of* $SL = \prod_{i=1}^{r-1} SL(V_i)$*, is*

$$SL(n, \mathbb{C})/[P, P],$$

where P *is the parabolic associated to the flag* $(V_1, \dots, V_r)$*.*

This statement also holds for quivers

$$0 \leftarrow V_1 \xleftarrow{\beta_1} V_2 \xleftarrow{\beta_2} \cdots \xleftarrow{\beta_{r-2}} V_{r-1} \xleftarrow{\beta_{r-1}} V_r = \mathbb{C}^n,$$

where the maps $\beta_i\colon V_{i+1} \to V_i$ *are surjective.*

Proof. We prove this for the case of β_i surjective, and the case of α_i injective follows by dualising.

We may choose bases for the V_i so that

$$\beta_i = (0_{n_i \times k_i} \mid I_{n_i \times n_i}),$$

where $n_i = \dim V_i$ and $k_i = n_{i+1} - n_i$ is the dimension of the kernel of β_i. Explicitly, once we have chosen such a basis $e_1, \dots, e_{n_i}$ for V_i, we can choose a basis e'_j, $1 \leqslant j \leqslant n_{i+1}$, for V_{i+1} so that the first $n_{i+1} - n_i$ elements are a basis for $\ker \beta_i$ and for the remaining n_i elements we have $\beta_i \colon e'_j \mapsto e_j$. We may view this as using the action of $SL \times SL(n, \mathbb{C}) = \prod_{i=1}^r SL(n_i, \mathbb{C})$ to standardise the β_i.

How much freedom do we have in choosing such bases? Since β_i transforms by $\beta_i \mapsto g_i \beta_i g_{i+1}^{-1}$, if β_i is in the above standard form, then $\beta_i = g_i \beta_i g_{i+1}^{-1}$ if and only if

$$g_{i+1} = \begin{pmatrix} * & * \\ 0 & g_i \end{pmatrix},$$

where the top left block is $k_i \times k_i$ and the bottom right is $n_i \times n_i$. Moreover g_1 is an arbitrary element of $SL(n_1, \mathbb{C})$. We see inductively that the freedom in $SL(n, \mathbb{C})$ is the commutator of the parabolic group P associated to the flag of dimensions $(n_1, n_2, \dots, n_r = n)$ in $\mathbb{C}^n$. $\square$

We conclude that the GIT quotient of the space of full flag quivers by SL gives us a description of the symplectic implosion for $SL(n, \mathbb{C})$.

Theorem 4.11. *The GIT quotient of the space of full flag quivers by $SL = \prod_{i=2}^{n-1} SL(i, \mathbb{C})$ is the symplectic implosion for $SU(n)$. The stratification by quiver diagrams as above corresponds to the stratification of the implosion as the disjoint union over the standard parabolic subgroups P of $SL(n, \mathbb{C})$ of the varieties $SL(n, \mathbb{C})/[P, P]$.*

Proof. We have already identified the strata of the GIT quotient of the space of full flag quivers with the strata of the implosion. Now observe that the complement of the open stratum $SL(n, \mathbb{C})/N$ (where N is the maximal unipotent, i.e. the commutator of the Borel subgroup) is of complex codimension strictly greater than one. The universal symplectic implosion and the GIT quotient of the space of full flag quivers therefore have the same coordinate ring $\mathcal{O}(SL(n, \mathbb{C}))^N$, and as they are both affine varieties they are now isomorphic. $\square$

We recall the embedding [9] of the symplectic implosion

$$K_{\mathbb{C}} /\!\!/ N \subset E,$$

where E is the direct sum of K-modules $E = \oplus V_\varpi$, and V_ϖ is the K-module with highest weight ϖ. We take the sum over a minimal generating set for the monoid of dominant weights. In our case $K = SU(n)$ so these are just the exterior powers of the standard representation $\mathbb{C}^n$. We denote a highest weight vector of V_ϖ by v_ϖ.

Now we can define a map from the space of quivers to the space $E = \bigoplus_{j=1}^{n-1} \wedge^j \mathbb{C}^n$ by sending α to the element of E with jth component

$$\wedge^j \left(\alpha_{n-1} \dots \alpha_{j+1} \alpha_j\right) \mathrm{vol}_j \in \wedge^j \mathbb{C}^n, \tag{4.12}$$

where vol_j denotes the standard generator of $\wedge^j \mathbb{C}^j$.

Note that under the action of $SL = \prod_{i=1}^{r-1} SL(i,\mathbb{C})$, the composition $\alpha_{n-1}\dots\alpha_{j+1}\alpha_j$ gets postmultiplied by g_j^{-1}; but $g_j \in SL(j,\mathbb{C})$, so the jth exterior power is invariant. Hence our map descends to the GIT quotient by SL, and thus gives an explicit isomorphism of the GIT quotient to its image $K_{\mathbb{C}}/\!\!/N$ in E.

As $\mathbb{C}^j = \ker\alpha_j \oplus \mathbb{C}^{m_j}$, and the restriction of α_j to $\mathbb{C}^{m_j}$ is injective, we see that $\wedge^j(\alpha_{n-1}\dots\alpha_{j+1}\alpha_j)$ is zero if and only if $m_j < j$; i.e. α_j is not injective. Now, from (4.8)–(4.9), we see that knowing for which indices this occurs determines the full sequence of m_j, hence which stratum we are in. So the 2^{n-1} strata correspond to the possibilities for which components of (4.12) are zero.

Recall also that the symplectic implosion may be realised as the closure $\overline{K_{\mathbb{C}}v}$, where $v = \sum v_\varpi$ is the sum of the highest weight vectors. Using the Iwasawa decomposition $K_{\mathbb{C}} = KAN$ and recalling that N fixes v, we see that $\overline{K_{\mathbb{C}}v} = K(\overline{T_{\mathbb{C}}v})$, the sweep under the compact group K of a toric variety $\overline{T_{\mathbb{C}}v}$. In terms of the quiver model, recall from above we may use the action of $SL(n,\mathbb{C}) \times \prod_{i=1}^{r-1} SL(m_i,\mathbb{C})$ to put the maps $\bar{\alpha}_i$ into a form where the only non-zero entries are the (j,j) terms for $j = 1,\dots,m_i$, and that these terms may be set to be arbitrary non-zero scalars. Taking all these scalars to be 1, we arrive at the stratification in Lemma 4.10. If instead we take all the scalars for $\bar{\alpha}_i$ to be equal to a non-zero scalar σ_i, then the freedom in putting $\bar{\alpha}_i$ into this form is the parabolic P rather than its commutator. This gives a description of the strata as $(\mathbb{C}^*)^{r-1}$-bundles over the *compact* flag variety $SL(n,\mathbb{C})/P$. This is of course just reflecting the fact that $P = [P,P].T_{\mathbb{C}}^{r-1}$. Fixing basepoints in the flag varieties and taking all the strata together gives the toric variety $\overline{T_{\mathbb{C}}v}$.

5. Hyperkähler quiver diagrams

We now turn our attention to hyperkähler quiver diagrams. For $K = SU(n)$ actions this gives us a finite-dimensional approach to constructing the universal hyperkähler implosion. This uses work on quiver varieties due to Nakajima [20] and Kobak and Swann [12] (see also Bielawski [1, 2]). We want to produce a hyperkähler stratified space with a torus action whose hyperkähler reductions by this action give Kostant varieties.

Choose integers $0 \leqslant n_1 \leqslant n_2 \leqslant \dots \leqslant n_r = n$ and consider the flat hyperkähler space

(5.1)
$$M = M(\mathbf{n}) = \bigoplus_{i=1}^{r-1} \mathbb{H}^{n_i n_{i+1}} = \bigoplus_{i=1}^{r-1} \operatorname{Hom}(\mathbb{C}^{n_i},\mathbb{C}^{n_{i+1}}) \oplus \operatorname{Hom}(\mathbb{C}^{n_{i+1}},\mathbb{C}^{n_i})$$

with the hyperkähler action of $U(n_1)\times\dots\times U(n_r)$

$$\alpha_i \mapsto g_{i+1}\alpha_i g_i^{-1}, \quad \beta_i \mapsto g_i\beta_i g_{i+1}^{-1} \qquad (i = 1,\dots r-1),$$

with $g_i \in U(n_i)$ for $i = 1, \ldots, r$. Here α_i and β_i denote elements of $\mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_{i+1}})$ and $\mathrm{Hom}(\mathbb{C}^{n_{i+1}}, \mathbb{C}^{n_i})$ respectively, and right quaternion multiplication is given by

$$(\alpha_i, \beta_i)\mathbf{j} = (-\beta_i^*, \alpha_i^*). \tag{5.2}$$

We may write $(\alpha, \beta) \in M(\mathbf{n})$ as a quiver diagram:

$$0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} \mathbb{C}^{n_1} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \mathbb{C}^{n_2} \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{r-2}}{\overset{\alpha_{r-2}}{\rightleftarrows}} \mathbb{C}^{n_{r-1}} \underset{\beta_{r-1}}{\overset{\alpha_{r-1}}{\rightleftarrows}} \mathbb{C}^{n_r} = \mathbb{C}^n,$$

where $\alpha_0 = \beta_0 = 0$. For brevity, we will often call such a diagram a quiver. If each β_i is zero we recover a symplectic quiver diagram.

Let $\tilde{H}$ be the subgroup, isomorphic to $\prod_{i=1}^{r-1} U(n_i)$, given by setting $g_r = 1$ and let

$$\tilde{\mu}\colon M \to \mathrm{Lie}(\tilde{H}) \otimes \mathbb{R}^3 = \mathrm{Lie}(\tilde{H}) \otimes (\mathbb{R} + \mathbb{C})$$
$$\tilde{\mu}(\alpha, \beta) = \Big((\alpha_i\alpha_i^* - \beta_i^*\beta_i + \beta_{i+1}\beta_{i+1}^* - \alpha_{i+1}^*\alpha_{i+1})\mathbf{i}, \alpha_i\beta_i - \beta_{i+1}\alpha_{i+1}\Big)$$

be the hyperkähler moment map. Hyperkähler quotients $\tilde{\mu}^{-1}(c)/\tilde{H}$ of M by $\tilde{H}$ (with $c \in \mathrm{Lie}(Z(\tilde{H})) \otimes \mathbb{R}^3$ where $Z(\tilde{H}) \cong T^{r-1}$ is the centre of $\tilde{H}$) will admit a residual hyperkähler action of $U(n_r) = U(n)$, although in fact only $SU(n)$ acts (almost) effectively as the diagonal central $U(1)$ acts trivially.

It is proved in [12] (see also [15]) that when we have a full flag (that is, when $r = n$ and $n_j = j$ for each j, so that the centre of $\tilde{H}$ can be identified with the maximal torus T of $K = SU(n)$) then the hyperkähler quotient $\tilde{\mu}^{-1}(0)/\tilde{H}$ of M by $\tilde{H}$ can be identified with the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$. Of course the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$ is the closure of a generic nilpotent coadjoint orbit. On the other hand it is proved in [20] that cotangent bundles of generalised flag varieties (which are diffeomorphic to semisimple orbits of $\mathfrak{sl}(n, \mathbb{C})$) may be obtained as hyperkähler quotients $\tilde{\mu}^{-1}(c)/\tilde{H}$ of M by $\tilde{H}$ for generic non-zero c.

Now we may instead reduce in stages by first reducing with respect to the group $H = \prod_{i=1}^{r-1} SU(n_i)$ to obtain a hyperkähler space $Q = M/\!\!/\!\!/H$, which has a residual action of the torus T^{r-1} as well as an action of $SU(n_r) = SU(n)$, with the hyperkähler quotients of Q by T^{r-1} coinciding with the hyperkähler quotients of M by $\tilde{H}$. This makes the follow definition reasonable.

Definition 5.3. The *universal hyperkähler implosion for* $SU(n)$ will be the hyperkähler quotient $Q = M/\!\!/\!\!/H$, where M, H are as above with $n_j = j$, for $j = 1, \ldots, n$, (i.e. the case of a full flag quiver).

Note that as Q is a hyperkähler reduction by H at level 0, it inherits an $SU(2)$ action that rotates the two-sphere of complex structures: this action is induced from multiplication by unit quaternions on $M = \mathbb{H}^{\sum_{i=1}^{r-1} n_i n_{i+1}}$ on the other side from that on which H acts and includes

the transformation (5.2). We denote these group actions on M, Q, etc., by

$$SU(2)_{\mathrm{rotate}}. \tag{5.4}$$

Let us take a more detailed look at the structure of Q, in the general case where the flag is not necessarily full.

The components of the complex moment map $\mu_{\mathbb{C}}$ for the H action on M are the trace-free parts of $\alpha_i\beta_i - \beta_{i+1}\alpha_{i+1}$ for $0 \leqslant i \leqslant r-2$, because we are performing a reduction by special unitary rather than unitary groups. Hence, the complex moment map equation $\mu_{\mathbb{C}} = 0$ can be expressed as the requirement that

$$\alpha_{i-1}\beta_{i-1} - \beta_i\alpha_i = \lambda_i^{\mathbb{C}} I \qquad (i = 1, \dots, r-1), \tag{5.5}$$

for some complex scalars $\lambda_1^{\mathbb{C}}, \dots, \lambda_{r-1}^{\mathbb{C}}$. Similarly the real moment map equation $\mu_{\mathbb{R}} = 0$ can be expressed as:

$$\alpha_{i-1}\alpha_{i-1}^* - \beta_{i-1}^*\beta_{i-1} + \beta_i\beta_i^* - \alpha_i^*\alpha_i = \lambda_i^{\mathbb{R}} I \quad (i = 1, \dots, r-1), \tag{5.6}$$

where $\lambda_i^{\mathbb{R}}$ are real scalars.

The hyperkähler quotient $Q = M /\!\!/\!\!/ H = (\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0))/H$ is the symplectic quotient of the affine variety $\mu_{\mathbb{C}}^{-1}(0) \subset M$ by the compact group H. The work of Kempf and Ness [10] (cf. [11, 22]) shows that Q can be canonically identified with the GIT quotient $\mu_{\mathbb{C}}^{-1}(0) /\!\!/ H_{\mathbb{C}}$ of $\mu_{\mathbb{C}}^{-1}(0)$ by the complexification

$$H_{\mathbb{C}} = \prod_{i=1}^{r-1} SL(n_i, \mathbb{C})$$

of H. This identification proceeds via the H-invariant composition

$$\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0) \to \mu_{\mathbb{C}}^{-1}(0) \to \mu_{\mathbb{C}}^{-1}(0) /\!\!/ H_{\mathbb{C}}$$

of the inclusion of $\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0)$ in $\mu_{\mathbb{C}}^{-1}(0)$ with the natural map $\mu_{\mathbb{C}}^{-1}(0) \to \mu_{\mathbb{C}}^{-1}(0) /\!\!/ H_{\mathbb{C}}$. The action of $H_{\mathbb{C}}$ is given by

$$\alpha_i \mapsto g_{i+1}\alpha_i g_i^{-1}, \quad \beta_i \mapsto g_i\beta_i g_{i+1}^{-1} \qquad (i = 1, \dots r-2),$$
$$\alpha_{r-1} \mapsto \alpha_{r-1} g_{r-1}^{-1}, \quad \beta_{r-1} \mapsto g_{r-1}\beta_{r-1},$$

where $g_i \in SL(n_i, \mathbb{C})$. Alternatively, one may first perform the hyperkähler quotient by taking the Kähler or GIT quotient $\mu_{\mathbb{R}}^{-1}(0)/H = M /\!\!/ H_{\mathbb{C}}$, and considering the level set cut out by the image of $\mu_{\mathbb{C}}^{-1}(0)$.

In this GIT picture, we have a residual action of $SL(n, \mathbb{C}) = SL(n_r, \mathbb{C})$ on the quotient Q given by

$$\alpha_{r-1} \mapsto g_r\alpha_{r-1}, \quad \beta_{r-1} \mapsto \beta_{r-1} g_r^{-1}.$$

Explicitly, we see that the action of the g_i (for $1 \leqslant i \leqslant r-1$) just conjugates the left-hand side of (5.5), so preserves the equations. This action commutes with the residual action of $\tilde{H}_{\mathbb{C}}/H_{\mathbb{C}}$ which we can identify with $(\mathbb{C}^*)^{r-1}$ (and with the maximal torus $T_{\mathbb{C}}$ of $K_{\mathbb{C}}$ in the case of a full flag, via the basis of $\mathfrak{t}$ given by the simple roots). Again, we note

that if all β_i are zero then the complex moment map equations hold trivially and we recover the symplectic quiver situation of §4.

For each quiver diagram $(\alpha, \beta) \in M(\mathbf{n})$, we define

$$X = \alpha_{r-1}\beta_{r-1} \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n).$$

This element is invariant under the action of $\tilde{H}_{\mathbb{C}}$ and transforms by conjugation under the residual $GL(n, \mathbb{C}) = GL(n_r, \mathbb{C})$ action. In fact, it is the complex-symplectic moment map for the action of $GL(n, \mathbb{C})$ on $M(\mathbf{n})$. We get an $(\mathbb{C}^*)^{r-1}$-invariant and $SL(n, \mathbb{C})$-equivariant map $Q \to \mathfrak{sl}(n, \mathbb{C})$ induced by:

$$(\alpha, \beta) \mapsto (X)_0 = X - \frac{1}{n}\,\mathrm{tr}(X) I_n \tag{5.7}$$

where I_n is the $n \times n$-identity matrix, which is the complex-symplectic moment map for the action of $SL(n, \mathbb{C})$ on $M(\mathbf{n})$.

We start by obtaining information on the eigenvalues of X and other endomorphisms derived from final segments of the quiver. Define

$$X_k = \alpha_{r-1}\alpha_{r-2}\dots\alpha_{r-k}\beta_{r-k}\dots\beta_{r-2}\beta_{r-1} \qquad (1 \leqslant k \leqslant r-1) \tag{5.8}$$

so that $X = X_1$.

Lemma 5.9. *For $(\alpha, \beta) \in \mu_{\mathbb{C}}^{-1}(0)$, satisfying* (5.5)*, we have*

$$X_k X = X_{k+1} - (\lambda^{\mathbb{C}}_{r-1} + \dots + \lambda^{\mathbb{C}}_{r-k}) X_k. \tag{5.10}$$

Proof. This is a straightforward consequence of the equations (5.5). We have

$$\begin{aligned} X_k X &= \alpha_{r-1}\dots\alpha_{r-k}\beta_{r-k}\dots\beta_{r-2}\beta_{r-1}\alpha_{r-1}\beta_{r-1} \\ &= \alpha_{r-1}\dots\alpha_{r-k}\beta_{r-k}\dots\beta_{r-2}(\alpha_{r-2}\beta_{r-2} - \lambda^{\mathbb{C}}_{r-1})\beta_{r-1} \\ &= \alpha_{r-1}\dots\alpha_{r-k}\beta_{r-k}\dots\beta_{r-2}\alpha_{r-2}\beta_{r-2}\beta_{r-1} - \lambda^{\mathbb{C}}_{r-1} X_k. \end{aligned}$$

We repeat this process, using the equations successively to shuffle the α term from X forward until it meets the other α's. Each such shuffle means we pick up a $-\lambda^{\mathbb{C}}_j X_k$ term. After k such operations, we have the desired result. □

Putting $\nu_i = \sum_{j=i}^{r-1} \lambda^{\mathbb{C}}_j$, so that $X_k(X+\nu_{r-k}) = X_{k+1}$, we find inductively

$$X(X + \nu_{r-1})\dots(X + \nu_1) = 0. \tag{5.11}$$

We thus have an annihilating polynomial for X in terms of the $\lambda^{\mathbb{C}}_i$; in particular, if all $\lambda^{\mathbb{C}}_i$ are zero, then X is nilpotent (cf. [12]).

To gain more information, we decompose $\mathbb{C}^n$ as the direct sum $\bigoplus_{j=1}^{\ell} \ker(X - \tau_j I)^{m_j}$, where

$$(x - \tau_1)^{m_1}\dots(x - \tau_\ell)^{m_\ell},$$

is the characteristic polynomial of X and the τ_j are distinct. More generally, for each i, we decompose

$$\mathbb{C}^{n_i} = \bigoplus_{j=1}^{\ell_i} \ker(\alpha_{i-1}\beta_{i-1} - \tau_{i,j} I)^{m_{ij}} \tag{5.12}$$

where the $\tau_{i,j}$ for $1 \leqslant j \leqslant \ell_i$ are the eigenvalues of $\alpha_{i-1}\beta_{i-1}$, with associated generalised eigenspaces given by the summands on the right hand side of (5.12).

Since $(\alpha, \beta) \in \mu_{\mathbb{C}}^{-1}(0)$, equation (5.5) shows that $\beta_i(\alpha_i\beta_i - \tau I) = (\beta_i\alpha_i - \tau I)\beta_i = (\alpha_{i-1}\beta_{i-1} - (\lambda_i^{\mathbb{C}} + \tau)I)\beta_i$, for any scalar τ. Thus

$$\beta_i(\alpha_i\beta_i - \tau I)^m = (\alpha_{i-1}\beta_{i-1} - (\lambda_i^{\mathbb{C}} + \tau)I)^m\beta_i,$$

for each m, and β_i restricts to a map

$$\beta_i \colon \ker(\alpha_i\beta_i - \tau I)^m \to \ker(\alpha_{i-1}\beta_{i-1} - (\lambda_i^{\mathbb{C}} + \tau)I)^m. \tag{5.13}$$

A similar calculation shows that α_i restricts to a map

$$\alpha_i \colon \ker(\alpha_{i-1}\beta_{i-1} - (\lambda_i^{\mathbb{C}} + \tau)I)^m \to \ker(\alpha_i\beta_i - \tau I)^m. \tag{5.14}$$

We obtain an endomorphism $\alpha_i\beta_i$ of $\ker(\alpha_i\beta_i - \tau I)^m$, which is an isomorphism unless $\tau = 0$. Similarly the composition $\beta_i\alpha_i$ is an isomorphism of $\ker(\alpha_{i-1}\beta_{i-1} - (\lambda_i^{\mathbb{C}} + \tau)I)^m = \ker(\beta_i\alpha_i - \tau I)^m$ onto itself unless $\tau = 0$. Therefore the maps (5.13) and (5.14) are bijective unless $\tau = 0$.

We deduce that $\tau \neq 0$ is an eigenvalue of $\alpha_i\beta_i$ if and only if $\tau + \lambda_i^{\mathbb{C}} \neq \lambda_i^{\mathbb{C}}$ is an eigenvalue of $\alpha_{i-1}\beta_{i-1}$. Moreover α_i and β_i define isomorphisms between the associated generalised eigenspaces. In addition, if the dimension vector $(n_1, \dots, n_r)$ is strictly ordered, $0 < n_1 < n_2 < \dots < n_r = n$, then $\alpha_i\beta_i \in \operatorname{End}(V_{i+1})$ has zero as an eigenvalue, and α_i, β_i restrict to maps between the associated generalised 0-eigenspace and the generalised eigenspace for $\alpha_{i-1}\beta_{i-1}$ associated to $\lambda_i^{\mathbb{C}}$ (this latter space may of course be zero).

This gives us the following lemma:

Lemma 5.15. *Suppose the dimension vector* $\mathbf{n}$ *is strictly ordered. Then for* $(\alpha, \beta) \subset M(\mathbf{n})$ *satisfying* (5.5)*, the trace-free endomorphism* $(X)_0 = (\alpha_{r-1}\beta_{r-1})_0$*, defined at equation* (5.7)*, has eigenvalues* $\kappa_1, \dots, \kappa_r$*, where*

$$\kappa_j = \frac{1}{n}\Big(n_1\lambda_1^{\mathbb{C}} + n_2\lambda_2^{\mathbb{C}} + \dots + n_{j-1}\lambda_{j-1}^{\mathbb{C}} \\ - (n - n_j)\lambda_j^{\mathbb{C}} - (n - n_{j+1})\lambda_{j+1}^{\mathbb{C}} - \dots - (n - n_{r-1})\lambda_{r-1}^{\mathbb{C}}\Big).$$

The eigenvalue κ_j *occurs with algebraic multiplicity at least* $n_j - n_{j-1}$*. If* $\kappa_1, \dots, \kappa_r$ *are all distinct the multiplicity of* κ_j *is exactly* $n_j - n_{j-1}$*.*

Moreover if $i \leqslant j$ *then*

$$\kappa_{j+1} - \kappa_i = \lambda_i^{\mathbb{C}} + \lambda_{i+1}^{\mathbb{C}} + \dots + \lambda_j^{\mathbb{C}}.$$

It follows from the argument above that we can use generalised eigenspaces to decompose our quiver as a direct sum of quivers with maps $\alpha_{i,j}, \beta_{i,j}$,

$$V_i^j \underset{\beta_{i,j}}{\overset{\alpha_{i,j}}{\rightleftarrows}} V_{i+1}^j,$$

satisfying $\alpha_{i,j}\beta_{i,j} - \beta_{i+1,j}\alpha_{i+1,j} = \lambda_{i+1}^{\mathbb{C}}$ and such that $\alpha_{i,j}\beta_{i,j}$ has only one eigenvalue $\tau_{i+1,j}$.

Suppose for some j we have that $\alpha_{k,j}$ and $\beta_{k,j}$ are isomorphisms for $i < k < s$ but not for $k = i$ or $k = s$. Then $\tau_{i+1,j} = 0$, $\tau_{s+1,j} = 0$ and Lemma 5.15 implies that $\sum_{k=i+1}^{s} \lambda_k^{\mathbb{C}} = 0$.

On the other hand, if $\tau_{i+1,j}$ is non-zero, then $\alpha_{i,j}$ and $\beta_{i,j}$ are isomorphisms.

Remark 5.16. Whenever $\alpha_{i,j}$ is an isomorphism and $i < r-1$, equation (5.5) implies that $\beta_{i,j} = (\alpha_{i,j})^{-1}(\lambda_{i+1}^{\mathbb{C}} + \beta_{i+1,j}\alpha_{i+1,j})$. We may now perform a contraction of the subquiver analogous to that in the symplectic case, by replacing

$$V_{i-1}^j \underset{\beta_{i-1,j}}{\overset{\alpha_{i-1,j}}{\rightleftarrows}} V_i^j \underset{\beta_{i,j}}{\overset{\alpha_{i,j}}{\rightleftarrows}} V_{i+1}^j \underset{\beta_{i+1,j}}{\overset{\alpha_{i+1,j}}{\rightleftarrows}} V_{i+2}^j$$

with

$$V_{i-1}^j \underset{\beta_{i-1,j}}{\overset{\alpha_{i-1,j}}{\rightleftarrows}} V_i^j \underset{(\alpha_{i,j})^{-1}\beta_{i+1,j}}{\overset{\alpha_{i+1,j}\alpha_{i,j}}{\rightleftarrows}} V_{i+2}^j.$$

The complex moment map equations for the contracted quiver are now satisfied with

$$\alpha_{i-1,j}\beta_{i-1,j} - (\alpha_{i,j})^{-1}\beta_{i+1,j}\alpha_{i+1,j}\alpha_{i,j} = \lambda_i^{\mathbb{C}} + \lambda_{i+1}^{\mathbb{C}}.$$

Conversely, given $\alpha_{i,j}$ and $\lambda_{i+1}^{\mathbb{C}}$ we may recover $\beta_{i,j}$ from the contracted quiver and reverse the process.

Observe that in the situation described above where $\alpha_{k,j}$ and $\beta_{k,j}$ are isomorphisms for $i < k < s$ but not for $k = i$ or $k = s$, then iterating the above procedure and using the relation $\sum_{k=i+1}^{s} \lambda_k^{\mathbb{C}} = 0$ implies (suppressing the j index) that $\alpha_i\beta_i\beta_{i+1}\dots\beta_{s-1} = \beta_{i+1}\dots\beta_s\alpha_s$.

Note that given an identification of V_{i+1}^j with V_i^j, we may apply the action of $SL(V_{i,j})$ to set $\alpha_{i,j}$ to be a non-zero scalar multiple aI of the identity. If we have a $GL(V_{i,j})$ action available we may set $a = 1$.

Example 5.17. Suppose a quiver

$$0 \rightleftarrows \mathbb{C}^m \rightleftarrows \mathbb{C}^m \rightleftarrows \cdots \rightleftarrows \mathbb{C}^m \rightleftarrows 0,$$

where there are p copies of $\mathbb{C}^m$, satisfies the complex moment map equations for $\prod_{k=1}^{p} SL(m,\mathbb{C})$, with $\sum_{k=1}^{p} \lambda_k^{\mathbb{C}} = 0$ but $\sum_{k=i}^{p} \lambda_k^{\mathbb{C}} \neq 0$ for $i > 1$, and contracts to the zero quiver

$$0 \rightleftarrows \mathbb{C}^m \rightleftarrows 0.$$

Then it lies in the $\prod_{k=1}^{p} SL(m,\mathbb{C})$-orbit of a quiver of the form

$$0 \rightleftarrows \mathbb{C}^m \underset{b_1}{\overset{a_1}{\rightleftarrows}} \mathbb{C}^m \rightleftarrows \cdots \rightleftarrows \mathbb{C}^m \underset{b_{p-1}}{\overset{a_{p-1}}{\rightleftarrows}} \mathbb{C}^m \rightleftarrows 0$$

where the maps $\mathbb{C}^m \rightleftarrows \mathbb{C}^m$ are multiplication by scalars a_j and b_j satisfying $a_j b_j = \sum_{k=j+1}^{p} \lambda_k^{\mathbb{C}}$. ♢

Remark 5.18. Suppose $\alpha_{r-1,j}$ is an isomorphism. Then we may use the complex moment map equation to write $\beta_{r-1,j} = (\alpha_{r-2,j}\beta_{r-2,j} - \lambda^{\mathbb{C}}_{r-1})\alpha^{-1}_{r-1,j}$. We may contract the right-hand end of the subquiver to get

$$\cdots \rightleftarrows V_{r-3,j} \underset{\beta_{r-3,j}}{\overset{\alpha_{r-3,j}}{\rightleftarrows}} V_{r-2,j} \underset{\beta_{r-2,j}(\alpha_{r-1,j})^{-1}}{\overset{\alpha_{r-1,j}\alpha_{r-2,j}}{\rightleftarrows}} V_{r,j}$$

satisfying the complex moment map equations at $(\lambda^{\mathbb{C}}_1, \dots, \lambda^{\mathbb{C}}_{r-2})$.

If we are primarily interested in quivers with strictly ordered dimension vector so that Lemma 5.15 applies, then Remarks 5.16 and 5.18 imply that for many purposes it is enough to consider the case when $\tau_{i+1,j} = 0$ for all i; that is, when $X = \alpha_{r-1}\beta_{r-1}$ is nilpotent and all $\lambda^{\mathbb{C}}_i$ are zero, so that the quiver satisfies the complex moment map equations $\tilde{\mu}_{\mathbb{C}} = 0$,

$$\alpha_i\beta_i = \beta_{i+1}\alpha_{i+1}, \tag{5.19}$$

for $\tilde{H}_{\mathbb{C}} = \prod_{i=1}^{r-1} GL(n_i, \mathbb{C})$. In this situation we have the following result from [12].

Proposition 5.20. *For any dimension vector* $\mathbf{n}$*, the orbit under the action of* $\tilde{H}_{\mathbb{C}} = \prod_{i=1}^{r-1} GL(n_i, \mathbb{C})$ *of a quiver*

$$0 = V_0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} V_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} V_2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{r-2}}{\overset{\alpha_{r-2}}{\rightleftarrows}} V_{r-1} \underset{\beta_{r-1}}{\overset{\alpha_{r-1}}{\rightleftarrows}} V_r = \mathbb{C}^n,$$

with $\dim V_j = n_j$*, which satisfies the complex moment map equations* (5.19) *for* $\tilde{H}_{\mathbb{C}}$*, is closed if and only if it is the direct sum of a quiver*

$$0 = V_0^{(*)} \underset{\beta_0^{(*)}}{\overset{\alpha_0^{(*)}}{\rightleftarrows}} V_1^{(*)} \underset{\beta_1^{(*)}}{\overset{\alpha_1^{(*)}}{\rightleftarrows}} V_2^{(*)} \underset{\beta_2^{(*)}}{\overset{\alpha_2^{(*)}}{\rightleftarrows}} \cdots \underset{\beta_{r-2}^{(*)}}{\overset{\alpha_{r-2}^{(*)}}{\rightleftarrows}} V_{r-1}^{(*)} \underset{\beta_{r-1}^{(*)}}{\overset{\alpha_{r-1}^{(*)}}{\rightleftarrows}} V_r^{(*)} = \mathbb{C}^n,$$

where $\alpha_j^{(*)}$ *is injective and* $\beta_j^{(*)}$ *is surjective for* $1 \leqslant j < r$ *(so for some* k*,* $V_j^{(*)} = 0$ *for* $0 \leqslant j \leqslant k$*) and a quiver*

$$0 = V_0^{(0)} \rightleftarrows V_1^{(0)} \rightleftarrows V_2^{(0)} \rightleftarrows \cdots \rightleftarrows V_{r-1}^{(0)} \rightleftarrows V_r^{(0)} = 0$$

in which all maps are 0.

Proof. The arguments of [12, Theorem 2.1] show that for $(\alpha, \beta) \in \tilde{\mu}_{\mathbb{C}}^{-1}(0)$ the closed $\tilde{H}_{\mathbb{C}}$-orbit condition corresponds to having direct sums $V_i = \ker\alpha_i \oplus \operatorname{im}\beta_i$. The complex moment map equations (5.19), imply that this is a direct sum decomposition into subquivers, so the maps are zero on the subquiver with $V_i^{(0)} = \ker\alpha_i$, and have the desired injectivity/surjectivity on the subquiver with $V_i^{(*)} = \operatorname{im}\beta_i$. □

Remark 5.21. Let us now decompose a strictly ordered quiver representation into a sum of subquivers determined by the generalised eigenspaces of the compositions $\alpha_i\beta_i$. These subquivers may be contracted as in Remarks 5.16 and 5.18 to shorter quivers with each $\lambda^{\mathbb{C}}_i = 0$.

The contracted subquivers thus satisfy the complex moment map equations (5.19) for the groups $\prod_i GL(V_{i,j})$ which correspond in their situations to $\tilde{H}_{\mathbb{C}}$. They lie in closed orbits for these groups which play the role of $\tilde{H}_{\mathbb{C}}$ *provided that* each subquiver lies in a closed orbit for the action of $\prod_i GL(V_{i,j})$. If this is the case, we may now apply Proposition 5.20 to the contracted subquivers and deduce, with the help of Example 5.17, that the original quiver is the direct sum of a quiver in which every α_i is injective and every β_i is surjective and quivers of the form

$$0 \rightleftarrows \mathbb{C}^m \underset{b_1}{\overset{a_1}{\rightleftarrows}} \mathbb{C}^m \underset{b_2}{\overset{a_1}{\rightleftarrows}} \cdots \underset{b_{p-2}}{\overset{a_{p-2}}{\rightleftarrows}} \mathbb{C}^m \underset{b_{p-1}}{\overset{a_{p-1}}{\rightleftarrows}} \mathbb{C}^m \rightleftarrows 0, \tag{5.22}$$

where the maps are multiplication by non-zero scalars a_j and b_j satisfying

$$a_j b_j = \sum_{k=j+1}^{p} \lambda_k^{\mathbb{C}}. \tag{5.23}$$

Unfortunately, if the original quiver lies in a closed $H_{\mathbb{C}}$-orbit but not a closed $\tilde{H}_{\mathbb{C}}$-orbit, we cannot deduce directly that the subquivers lie in closed orbits for the action of $\prod_i GL(V_{i,j})$. However we will see that we can get around this difficulty by making suitable choices of complex structures.

Remark 5.24. Note that for a quiver of the form (5.22) we may use the $GL(m, \mathbb{C})^p$ action to set the $a_i = 1$, so the b_i are now determined by the equations. Such a quiver will be left invariant by any $g \in GL(m, \mathbb{C})^p$ with $g_1 = \cdots = g_p \in GL(m, \mathbb{C})$, and hence for each such summand we will pick up a residual circle action on the injective/surjective quiver.

We introduce some notation that will also be useful later for describing an augmentation process for quivers.

Definition 5.25. Let S be a relation on $\{1, \ldots, r-1\}$. This is the same as a subset of $\{1, \ldots, r-1\} \times \{1, \ldots, r-1\}$.

Such an S is a *subrelation* of another relation S' if $(i, j) \in S$ implies $(i, j) \in S'$ for all $i, j \in \{1, \ldots, r-1\}$.

Note that any S defines a subrelation $\leqslant_S$ of $\leqslant$ by

$$\leqslant_S = \{(i, j) \in S : i \leqslant j\}$$

This is the maximal subrelation of $\leqslant$ contained in S.

Definition 5.26. To any relation S on $\{1, \ldots, r-1\}$ we associate the subtorus T_S of $\tilde{T} = T^{r-1} = \mathbb{R}^{r-1}/\mathbb{Z}^{r-1}$ whose Lie algebra is $\mathfrak{t}_S = \mathrm{Span}\{e_{ij} = \sum_{k=i}^{j} e_k : i \leqslant_S j\}$. We have an exact sequence

$$1 \longrightarrow H \longrightarrow \tilde{H} \xrightarrow{\varphi} \tilde{T} \longrightarrow 1,$$

where the ith component of φ is the determinant map $U(n_i) \to S^1$. We define H_S to be the pre-image

$$H_S = H_S(\mathbf{n}) = \varphi^{-1}(T_S).$$

In particular, $H_\varnothing = H$ and $H_\leqslant = \tilde{H}$.

Given any $\lambda = (\lambda_1, \dots, \lambda_{r-1}) \in (\mathbb{R}^3)^{r-1}$, we define a relation $\leqslant_\lambda$ on $\{1, \dots, r-1\}$ by

$$i \leqslant_\lambda j \iff (i \leqslant j \text{ and } \sum_{k=i}^{j} \lambda_k = 0 \text{ in } \mathbb{R}^3).$$

Proposition 5.27. *Suppose $(\alpha, \beta) \in M(\mathbf{n})$ satisfies the hyperkähler moment map equations (5.5)–(5.6) for H. Assume that $\mathbf{n}$ is strictly ordered. Then the group $SU(2)_{\mathrm{rotate}}$ of equation (5.4) contains an element that moves (α, β) to a quiver that is a direct sum of subquivers, one, $(\tilde{\alpha}, \tilde{\beta})$, with all $\tilde{\alpha}$'s injective and all $\tilde{\beta}$'s surjective, and the others of the scalar form (5.22), with a_i, b_i non-zero satisfying (5.23).*

Proof. Let $\lambda_j = (\lambda_j^{\mathbb{R}}, \lambda_j^{\mathbb{C}}) \in \mathbb{R} \times \mathbb{C} = \mathbb{R}^3$ be the values from the hyperkähler moment map equations. Then

$$(\lambda_1, \dots, \lambda_{r-1}) \in (\mathbb{R}^3)^{r-1} \cong \mathfrak{t}^{r-1} \otimes \mathbb{R}^3$$

can be identified with the value of the hyperkähler moment map for the action of the centre $T^{r-1} = Z(\tilde{H})$ of $\tilde{H}$.

The group $SU(2)_{\mathrm{rotate}}$ acts on λ_k as an $SO(3)$-rotation. Applying a generic element we may therefore ensure that

$$i \leqslant_\lambda j \iff (i \leqslant j \text{ and } \sum_{k=i}^{j} \lambda_k^{\mathbb{C}} = 0). \tag{5.28}$$

This implies that the rotated quiver (α, β) satisfies the full hyperkähler moment map equations (at level 0) for the subgroup $H_{\leqslant_\lambda}$ of $\tilde{H}$. Its orbit under the action of the complexification $(H_{\leqslant_\lambda})_{\mathbb{C}}$ is thus closed. Decomposing with respect to generalised eigenspaces and arguing as in Remark 5.21, we see that each contracted subquiver is closed under the action of $\prod_i GL(V_{i,j})$. By Remark 5.21, these subquivers have the claimed form. □

Now we have seen that injective/surjective quivers are relevant, we make the following observation about stabilisers:

Lemma 5.29. *If a quiver $(\alpha, \beta) \in M(\mathbf{n})$ has the property that for all i, either α_i is injective or β_i is surjective, then the stabiliser for the $\tilde{H}_{\mathbb{C}}$ action is trivial.*

Proof. If $g \in \prod_{i=1}^{r-1} GL(n_i, \mathbb{C})$ stabilises the quiver, then we have

$$g_{i+1}\alpha_i = \alpha_i g_i, \qquad g_i \beta_i = \beta_i g_{i+1},$$

for $i = 1, \dots, r-2$, with $\alpha_{r-1} = \alpha_{r-1} g_{r-1}$ and $g_{r-1}\beta_{r-1} = \beta_{r-1}$. Our injectivity/surjectivity assumption now means we can work inductively down from the top of the quiver to deduce each g_i is the identity element. □

Lemma 5.30. *Let S be any relation on $\{1, \dots, r-1\}$. Suppose that a quiver $(\alpha, \beta) \in M(\mathbf{n})$ has each α_i injective and each β_i surjective. Then it is stable in the sense of GIT for the action of the complexification $(H_S)_{\mathbb{C}}$ of H_S.*

In the case $H_S = H$, this is true under the weaker assumption that all α_i are injective or all β_i surjective.

Proof. Note that, by the argument of Lemma 5.29, the stabiliser in $\tilde{H}_{\mathbb{C}} = \prod_{i=1}^{r-1} GL(n_i, \mathbb{C})$ is trivial. Moreover for $1 \leqslant k < r$ the maps

$$\begin{gathered}\wedge^{n_k}(\alpha_{r-1}\alpha_{r-2}\dots\alpha_k)\colon\ \wedge^{n_k}\mathbb{C}^{n_k} \to \wedge^{n_k}\mathbb{C}^n \quad \text{and}\\ \wedge^{n_k}(\beta_k\dots\beta_{r-2}\beta_{r-1})\colon\ \wedge^{n_k}\mathbb{C}^n \to \wedge^{n_k}\mathbb{C}^{n_k}\end{gathered}$$

are both non-zero, and under the action of $(H_S)_{\mathbb{C}}$ one of them is multiplied by a character, while the other is multiplied by its inverse. This means that these maps are non-zero for every quiver in the closure of the $(H_S)_{\mathbb{C}}$-orbit, and hence that every quiver in the closure of the $(H_S)_{\mathbb{C}}$-orbit also has each α_i injective and each β_i surjective, and hence has trivial stabiliser. So the orbit is closed of maximal dimension, as required.

If $H_S = H$, this argument works under the weaker assumption as the exterior power maps are all invariant under the $H_{\mathbb{C}}$ action. □

Returning to the endomorphism $X = \alpha_{r-1}\beta_{r-1}$ associated to a quiver (α, β), we now note that (5.11) is actually the minimum polynomial of X if we impose appropriate non-degeneracy conditions on the quiver diagram.

Proposition 5.31. *Suppose $\mathbf{n} \in \mathbb{Z}^r_{>0}$ is strictly ordered and $(\alpha, \beta) \in M(\mathbf{n})$ satisfies the complex moment map equations (5.5) for $H_{\mathbb{C}}$. If each α_i is injective and each β_i surjective, then no polynomial of degree less than r annihilates X, and hence*

$$x(x + \nu_{r-1}) \dots (x + \nu_1)$$

is the minimum polynomial of X.

Proof. It follows from our formula (5.10) that if $p(x)$ is a degree k monic polynomial then $p(X)$ is a linear combination of X_k, $X_{k-1}, \dots$, $X_1 = X$ and I, with the coefficient of X_k being 1. We write $p(X) = X_k + c_{k-1}X_{k-1} + \dots + c_1 X + c_0 I$.

We may choose vectors w_i in V_r for $i = 2, \dots, r$ such that, for each i, the vector w_i is killed by $\beta_{i-1}\beta_i \dots \beta_{r-1}$ but not by $\beta_i \dots \beta_{r-1}$. To see this, let x_i be a non-zero element of $\ker \beta_{i-1}$ and then, using surjectivity of the β_j, let w_i satisfy $\beta_i \dots \beta_{r-1} w_i = x_i$.

If the α_i are injective, then, recalling the definition (5.8) of X_k, we see that X_j kills w_i if and only if $j \geqslant r - i + 1$.

Now if $k \leqslant r - 1$ and

$$p(X) = X_k + c_{k-1}X_{k-1} + \cdots + c_1 X + c_0 = 0$$

then successively applying this equation to $w_r, w_{r-1}, \ldots, w_2$ yields that each of $c_0, c_1, \ldots, c_{k-1}$ is zero. Hence X_k is zero, which is impossible since (using injectivity of α_i and surjectivity of β_i) we have that the rank of X_k equals $n_{r-k} = \dim V_{r-k}$. □

Actually we can get all the coadjoint orbits by considering quivers of this type.

Proposition 5.32. *Every element of $\mathfrak{sl}(n, \mathbb{C})$ may be obtained from a quiver*

$$0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} \mathbb{C}^{n_1} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \mathbb{C}^{n_2} \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{r-2}}{\overset{\alpha_{r-2}}{\rightleftarrows}} \mathbb{C}^{n_{r-1}} \underset{\beta_{r-1}}{\overset{\alpha_{r-1}}{\rightleftarrows}} \mathbb{C}^{n_r} = \mathbb{C}^n$$

with $0 < n_1 < n_2 < \cdots < n_r = n$, all α_i injective, all β_i surjective and satisfying the complex moment map equations (5.5) *for $H_{\mathbb{C}}$.*

Proof. Observe first of all that every element in $\mathfrak{sl}(n, \mathbb{C})$ may be realised as the trace-free part $(X)_0$ of some X in $\mathfrak{gl}(n, \mathbb{C})$ with a zero eigenvalue: if λ is an eigenvalue of $Y \in \mathfrak{sl}(n, \mathbb{C})$ then we can take $X = Y - \lambda I$. It is enough therefore to show that each $X \in \mathfrak{gl}(n, \mathbb{C})$ with a zero eigenvalue may be obtained from a quiver of the desired form. We prove this by induction on n.

Let X be such an element of $\mathfrak{gl}(n, \mathbb{C})$; then put $m = \operatorname{rank} X$ and choose an isomorphism $\phi\colon \operatorname{im} X \to \mathbb{C}^m$. Then taking $\beta = \phi \circ X\colon \mathbb{C}^n \to \mathbb{C}^m$ and $\alpha = \phi^{-1}\colon \mathbb{C}^m \to \operatorname{im} X \hookrightarrow \mathbb{C}^n$ we obtain $X = \alpha\beta$ with α injective and β surjective.

Let $Y = \beta\alpha \in \mathfrak{gl}(m, \mathbb{C})$ and pick an eigenvalue μ of Y. By the inductive hypothesis there is a quiver diagram in some $M(0 < n_1 < \cdots < n_{r-1} = m)$ with all α_i injective, all β_i surjective, satisfying the complex moment map equations for $\prod_{i=1}^{r-2} SL(n_i, \mathbb{C})$ and such that $\alpha_{r-2}\beta_{r-2} = Y - \mu I$. This equation gives $\alpha_{r-2}\beta_{r-2} - \beta\alpha = -\mu I$, so putting $\alpha_{r-1} = \alpha$, $\beta_{r-1} = \beta$ we obtain a quiver of the required form. □

6. Stratification for quiver diagrams

Let $\mathbf{n} = (0 = n_0 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r = n)$ and consider the space $M = M(\mathbf{n})$, defined at equation (5.1), of hyperkähler quiver diagrams

$$0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} \mathbb{C}^{n_1} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \mathbb{C}^{n_2} \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{r-2}}{\overset{\alpha_{r-2}}{\rightleftarrows}} \mathbb{C}^{n_{r-1}} \underset{\beta_{r-1}}{\overset{\alpha_{r-1}}{\rightleftarrows}} \mathbb{C}^{n_r} = \mathbb{C}^n. \tag{6.1}$$

Given a relation S on $\{1, \ldots, r-1\}$, we wish to describe the structure of $Q_S = M /\!\!/\!\!/ H_S$, where $H_S = \varphi^{-1}(T_S)$ is the subgroup of $\tilde{H} = \prod_{i=1}^{r-1} U(n_i)$ specified in Definition 5.26.

Definition 6.2. A quiver diagram $(\alpha, \beta) \in M$ will be called *hyperkähler stable* if, after applying some element of the group $SU(2)_{\mathrm{rotate}}$ defined in (5.4), each α_i is injective and each β_i is surjective.

The hyperkähler stable quivers form an open subset M^{hks} of M, and, by Lemma 5.29, H_S acts freely on the intersection of M^{hks} with the solution space of the hyperkähler moment map equations (since H_S is contained in its complexification with respect to any complex structure, and the definition of hyperkähler stable means that for each quiver in M^{hks} we may choose a complex structure to which Lemma 5.29 applies). It follows that the hyperkähler quotient $M^{\mathrm{hks}} /\!\!/\!\!/ H_S$ is an open subset Q_S^{hks} of $Q_S = M /\!\!/\!\!/ H_S$.

Lemma 6.3. *Q_S^{hks} is a hyperkähler manifold.*

Proof. Since Q_S^{hks} is an open subset of a hyperkähler quotient by H_S it is enough to check that it is non-singular. This follows from the freeness of the H_S action on M^{hks}. □

Remark 6.4. The non-singularity result in Lemma 6.3 also follows from Lemmas 5.29 and 5.30 since (by the work of Kempf and Ness [10, 22]) for any choice of complex structures we can identify Q_S^{hks} locally with the GIT quotient by $(H_S)_{\mathbb{C}}$ of the affine subvariety of the affine space M defined by the complex moment map equations. Moreover the $(H_S)_{\mathbb{C}}$ action is free on the neighbourhood where the identification takes place, and the subvariety is non-singular at any point whose stabiliser in $(H_S)_{\mathbb{C}}$ is trivial (or even finite).

Lemma 6.5. *Suppose that, after applying some element of the group $SU(2)_{\mathrm{rotate}}$ a quiver has the form* (6.1) *with each α_i injective. Then it is hyperkähler stable.*

Proof. If a quiver (α, β) has all α_i injective, then the same is true for all but finitely many of the quivers in its orbit under the action of $SU(2)_{\mathrm{rotate}}$. Now, the $SU(2)_{\mathrm{rotate}}$-action includes the right-multiplication by $\mathbf{j}$ given in (5.2). We conclude that all but finitely elements of the $SU(2)_{\mathrm{rotate}}$-orbit have β^* injective, which is the same as saying that β is surjective. Thus the quiver is hyperkähler stable. □

For the stratification results, we will describe an augmentation process for quiver diagrams. Note that $M(1,1) = \mathbb{H}$ with $a + \mathbf{j}b$ corresponding to the quiver $\mathbb{C} \underset{b}{\overset{a}{\rightleftarrows}} \mathbb{C}$, whose maps are multiplication by a and b, respectively. Given any integer $d > 0$ and indices $1 \leqslant i \leqslant j < r$, we have the element $e_{ij} = \sum_{k=i}^{j} e_k \in \mathbb{Z}^r \subset \mathbb{R}^r$, as in Definition 5.26, and may use the multiple $d\, e_{ij}$ as a dimension vector. For $p = j - i - 1$,

we define a hyperkähler embedding

$$\phi_{i,j;d}\colon \mathbb{H}^p \longrightarrow M(d\, e_{ij})$$

$$\phi_{i,j;d}(a+\mathbf{j}b) = \Bigl(0 \underset{0}{\overset{0}{\rightleftarrows}} \cdots \underset{0}{\overset{0}{\rightleftarrows}} 0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C}^d \underset{b_1}{\overset{a_1}{\rightleftarrows}} \mathbb{C}^d \underset{b_2}{\overset{a_2}{\rightleftarrows}} \cdots \underset{b_p}{\overset{a_p}{\rightleftarrows}} \mathbb{C}^d \underset{0}{\overset{0}{\rightleftarrows}} 0 \cdots \underset{0}{\overset{0}{\rightleftarrows}} 0\Bigr)$$

with each map in the quiver being multiplication by the indicated scalar (the sets on both sides are empty if $p = -1$). This map is an isomorphism for $d = 1$, and in all cases is equivariant with respect to the $SU(2)_{\mathrm{rotate}}$-action. Our construction will involve stacking the above embeddings on top of a quiver in $M(\mathbf{n})$.

Definition 6.6. A relation S on $\{1,\dots,r-1\}$ is said to be *injective* if S is the graph of an injective function, also denoted by S, from $\operatorname{dom} S = \{i \mid \exists j : (i,j) \in S\}$ to $\{1,\dots,r-1\}$.

We note that any subrelation of an injective relation is again injective.

Definition 6.7. Let S be an injective subrelation of $\leqslant$ on $\{1,\dots,r-1\}$. Suppose $\mathbf{m},\mathbf{n} \in \mathbb{Z}^r_{\geqslant 0}$ are two ordered dimension vectors and there is a function $\delta\colon \operatorname{dom} S \to \mathbb{Z}_{>0}$ such that

$$\mathbf{n} = \mathbf{m} + \mathbf{d},$$

for $\mathbf{d} = \sum_{(i,j)\in S} \delta(i) e_{ij}$. Put $R = <_S = \{(i,j) \in S : i < j\}$ and $\ell = \sum_{(i,j)\in R} j - i - 1$.

The *augmentation map* is the $SU(2)_{\mathrm{rotate}}$-equivariant hyperkähler embedding

$$\phi_{S,\delta}\colon M(\mathbf{m}) \oplus \mathbb{H}^\ell \to M(\mathbf{n})$$

obtained by writing $\mathbb{H}^\ell = \bigoplus_{(i,j)\in R} M(e_{ij})$ and mapping

$$((\alpha,\beta), (a^{(i)} + \mathbf{j}b^{(i)})_{i\in \operatorname{dom} R}) \text{ to } (\alpha,\beta) \oplus 0 \oplus \bigoplus \phi_{i,S(i);\delta(i)}(a^{(i)} + \mathbf{j}b^{(i)}),$$

where $0 \in M(\sum_{(i,i)\in S} d(i) e_i)$.

Recall now that the hyperkähler modification [6] of a hyperkähler manifold Y with a tri-Hamiltonian circle action is $Y_{\mathrm{mod}} = (Y \times \mathbb{H}) /\!/\!/ S^1$, where S^1 acts diagonally on $Y \times \mathbb{H}$ with respect to the given action on Y and the inverse of the standard action on $\mathbb{H}$. This construction adjusts to other choices of weight for the circle action on $\mathbb{H}$. More generally if a torus T^ℓ acts on Y with a hyperkähler moment map then we have a hyperkähler modification

$$\hat{Y} = (Y \times \mathbb{H}^\ell) /\!/\!/ T^\ell.$$

This is a hyperkähler space of the same dimension as Y that contains a copy of the hyperkähler quotient $Y /\!/\!/ T^\ell$.

In the situation of Definition 6.7 above, suppose $S \subset S_1$, where S_1 is also an injective subrelation of $\leqslant$. Associated to S_1 we have the subgroup $H_{S_1} = H_{S_1}(\mathbf{m})$ of $\tilde{H}(\mathbf{m}) = \prod_{j=1}^{r-1} U(m_j)$ from Definition 5.26.

Put $Q_1 = M(\mathbf{m}) /\!\!/\!\!/ H_{S_1}$. Then there is an action of $\tilde{H}(\mathbf{m})$ on $M(\mathbf{m})$ and an induced action of $\tilde{T} = \tilde{H}(\mathbf{m})/H(\mathbf{m})$ on Q_1 since H_{S_1} contains $H(\mathbf{m})$ as a normal subgroup.

Embed $T^\ell = \mathbb{R}^\ell/\mathbb{Z}^\ell$ into $\tilde{T}$ via $e_{(i,j);k} \mapsto e_{i+k,j}$ for $(i,j) \in S$ with $j > i$ and $k = 1, \dots, j-i$. We may now use the induced action of T^ℓ on Q_1 to construct a hyperkähler modification

$$\hat{Q}_1 = (Q_1 \times \mathbb{H}^\ell) /\!\!/\!\!/ T^\ell$$

and consider the open subset

$$\hat{Q}_1^{\mathrm{hks}} = \Big(Q_1^{\mathrm{hks}} \times (\mathbb{H} \setminus \{0\})^\ell\Big) /\!\!/\!\!/ T^\ell. \tag{6.8}$$

Proposition 6.9. *Suppose S_1 is an injective subrelation of $\leqslant$ on $\{1, \dots, r-1\}$ with $S_1 = S \coprod S_2$, a disjoint union. Let $\mathbf{m}$, $\mathbf{n}$, δ, $R = <_S$ and ℓ be as in Definition 6.7, with the additional assumption that $m_r = n = n_r$, and write $\phi_{S,\delta}$ for the resulting augmentation map.*

Put $Q_1 = M(\mathbf{m}) /\!\!/\!\!/ H_{S_1}$, $Q_2 = M(\mathbf{n}) /\!\!/\!\!/ H_{S_2}$ and let $\hat{Q}_1^{\mathrm{hks}}$ be the open subset of the hyperkähler modification $\hat{Q}_1$ as in equation (6.8). Then the augmentation map $\phi_{S,\delta}$ induces an augmentation map

$$\Phi_{S,\delta} \colon \hat{Q}_1 \to Q_2$$

of hyperkähler quotients which is $SU(2)_{\mathrm{rotate}}$-equivariant and an embedding of the smooth manifold $\hat{Q}_1^{\mathrm{hks}}$.

Proof. A point of $\hat{Q}_1$ is represented by $\mathbf{q} = ((\alpha, \beta), (a^{(i)} + \mathbf{j} b^{(i)})_{i \in \operatorname{dom} R})$ satisfying

(i) the H_{S_1}-hyperkähler moment map equations

$$\begin{gathered} \alpha_{i-1}\beta_{i-1} - \beta_i\alpha_i = \lambda_i^{\mathbb{C}} I, \\ \alpha_{i-1}\alpha_{i-1}^* - \beta_{i-1}^*\beta_{i-1} + \beta_i\beta_i^* - \alpha_i^*\alpha_i = \lambda_i^{\mathbb{R}} I, \end{gathered} \tag{6.10}$$

for $i = 1, \dots, r-1$, and for some $(\lambda^{\mathbb{R}}, \lambda^{\mathbb{C}})$ satisfying $\sum_{k=i}^j \lambda_k^{\mathbb{C}} = 0 = \sum_{k=i}^j \lambda_k^{\mathbb{R}}$, for all $(i,j) \in S_1$,

(ii) the T^ℓ-hyperkähler moment map equations

$$\begin{gathered} a_{k-i}^{(i)} b_{k-i}^{(i)} = \lambda_k^{\mathbb{C}} + \lambda_{k+1}^{\mathbb{C}} + \dots + \lambda_j^{\mathbb{C}}, \\ |a_{k-i}^{(i)}|^2 - |b_{k-i}^{(i)}|^2 = \lambda_k^{\mathbb{R}} + \lambda_{k+1}^{\mathbb{R}} + \dots + \lambda_j^{\mathbb{R}}. \end{gathered}$$

for $(i,j) \in R$ and $i < k \leqslant j$.

The quiver $(\alpha', \beta') = \phi_{S,\delta}(\mathbf{q})$ satisfies the analogue of the equations (6.10) with the same $(\lambda^{\mathbb{R}}, \lambda^C)$. In particular, since $S_2 \subset S_1$, (α', β') satisfies the H_{S_2}-hyperkähler moment map equations. Furthermore $H_{S_1} \times T^\ell$ acts on (α', β') as a subgroup of H_{S_2}. We thus have a well-defined, and $SU(2)_{\mathrm{rotate}}$-equivariant, map $\Phi_{S,\delta}$, as claimed.

For $\hat{Q}_1^{\mathrm{hks}}$, we can move our quiver via the $SU(2)_{\mathrm{rotate}}$ action so each α_i is injective, each β_i is surjective and all $a_{k-i}^{(i)}$ and $b_{k-i}^{(i)}$ are non-zero. In this case, we can recover a point in $\hat{Q}_1^{\mathrm{hks}}$ from the H_{S_2}-orbit of the

quiver $(\alpha', \beta') = \phi_{S,\delta}(\mathbf{q}) \in M(\mathbf{n})$ by restricting maps in (α', β') to the images of compositions of other maps in (α', β'), since the ambiguity in this process is exactly the action of $H_{S_1} \times T^\ell$, cf. Remark 5.24. Thus $\Phi_{S,\delta}$ is injective on $\hat{Q}_1^{\mathrm{hks}}$. $\square$

Remark 6.11. We note that $\hat{Q}_1^{\mathrm{hks}}$ determines Q_1^{hks} as a hyperkähler manifold. Both spaces are the base of torus fibrations from a common total space contained in $M(\mathbf{m}) \times \mathbb{H}^\ell$. The projection to $\hat{Q}_1^{\mathrm{hks}}$ is a Riemannian submersion, whereas that to Q_1^{hks} is just projection on to the $M(\mathbf{m})$ components. The pull-backs of the two metrics differ by scalings along the quaternionic directions of the torus actions. A description of this in terms of the twist construction may be found in [21].

Remark 6.12. The image $\Phi_{S,\delta}(\hat{Q}_1) \subset Q_2$ determines the original data $\mathbf{m}$, S and d as follows.

Choose a point in the image represented by a quiver (α', β') of largest possible rank. Then $m_i = \operatorname{rank} \beta'_i \beta'_{i+1} \dots \beta'_{r-1}$ and the corresponding quiver (α, β) obtained by restriction has α_i injective and β_i surjective.

As S is injective the function δ is determined by its values on the range of S. We determine S and δ recursively. Suppose we have found $S' \subset S$ and the corresponding values of δ. Put $\mathbf{m}' = \mathbf{m} + \sum_{(i,j) \in S'} \delta(i) e_{ij}$. The largest element of the range of $S \setminus S'$ is the largest j for which $m'_j < n_j$, the corresponding value of δ is $n_j - m'_j$. Now $j = S(i)$, where $i - 1 < j$ is the largest index such that $\operatorname{rank} \beta'_{i-1} \beta'_i \dots \beta'_{j-1}$ is strictly less than $(n_j - m'_j) + \operatorname{rank} \beta_{i-1} \beta_i \dots \beta_{j-1}$.

Theorem 6.13. *Let $\mathbf{n} = (n_1 < \dots < n_r = n)$ be strictly ordered and let $M(\mathbf{n})$ be the space of hyperkähler quiver diagrams* (5.1)*.*

Suppose S is an injective subrelation of $\leqslant$ on $\{1, \dots, r-1\}$. Let $\delta \colon \operatorname{dom} S \to \mathbb{Z}_{>0}$ be a function such that $\mathbf{m} = \mathbf{n} - \mathbf{d}$, $\mathbf{d} = \sum_{(i,j) \in S} \delta(i) e_{ij}$, is an ordered dimension vector. Put $Q_S = M(\mathbf{m}) /\!/\!/ H_S$. Then

$$Q_{(S,\delta)} = \Phi_{S,\delta}(\hat{Q}_S^{\mathrm{hks}})$$

is a smooth hyperkähler manifold that is a locally closed subset of $Q = M(\mathbf{n}) /\!/\!/ H$.

Furthermore,

$$Q = \coprod_{S,\delta} Q_{(S,\delta)}$$

is the disjoint union over all such choices of S and δ.

Proof. By Proposition 6.9, $Q_{(S,\delta)}$ is a smooth hyperkähler manifold. It is open in its closure, which is just $\Phi_{S,\delta}(\hat{Q}_S)$. Remark 6.12 implies that $Q_{(S,\delta)} \cap Q_{(S',\delta')} = \varnothing$ if $(S, \delta) \neq (S', \delta')$. Finally it follows from Proposition 5.27, that every quiver satisfying the hyperkähler moment map equations for H lies in some $Q_{(S,\delta)}$. $\square$

Remark 6.14. When S is empty, so that δ is empty, we have $Q_{(S,\delta)} = Q^{\mathrm{hks}}$.

Let us now consider the full flag case when $r = n$ and $n_i = i$ for $i \leqslant n$, so that $Q = M /\!\!/\!\!/ H$ is the universal hyperkähler implosion for $SU(n)$. We specify (S, δ), by listing the elements of S, ordered by the first component, followed by the corresponding values of δ.

When $n = 2$ there are two strata, $Q_{(\varnothing,\varnothing)} = Q^{\mathrm{hks}}$ and $Q_{(\{(1,1)\},1)}$ which consists of the zero quiver constructed as the direct sum of $0 \rightleftarrows 0 \rightleftarrows \mathbb{C}^2$ and $0 \rightleftarrows \mathbb{C} \rightleftarrows 0$.

When $n = 3$ the possible injective subrelations S of $\leqslant$ on $\{1, 2\}$ are $\varnothing$, the singletons $\{(1, 1)\}$, $\{(2, 2)\}$, $\{(1, 2)\}$, and the subset $\{(1, 1), (2, 2)\}$. The strata are as follows:

(i) $Q_{(\varnothing,\varnothing)} = Q^{\mathrm{hks}}$;

(ii) $Q_{(\{(1,1)\},1)}$ with elements given by the direct sum of a hyperkähler stable quiver $0 \rightleftarrows 0 \rightleftarrows \mathbb{C}^2 \rightleftarrows \mathbb{C}^3$ and the zero quiver $0 \rightleftarrows \mathbb{C} \rightleftarrows 0 \rightleftarrows 0$;

(iii) $Q_{(\{(2,2)\},1)}$ with elements given by the direct sum of a hyperkähler stable quiver $0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^3$ and the zero quiver $0 \rightleftarrows 0 \rightleftarrows \mathbb{C} \rightleftarrows 0$;

(iv) $Q_{(\{(1,2)\},1)}$ with elements given by the direct sum of a hyperkähler stable quiver $0 \rightleftarrows 0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^3$ and a quiver of the form $0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C} \rightleftarrows 0$ where the maps $\mathbb{C} \rightleftarrows \mathbb{C}$ are isomorphisms;

(v) $Q_{(\{(1,1),(2,2)\},(1,1))}$ with elements given by the direct sum of a hyperkähler stable quiver $0 \rightleftarrows 0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^3$ and the zero quivers $0 \rightleftarrows \mathbb{C} \rightleftarrows 0 \rightleftarrows 0$ and $0 \rightleftarrows 0 \rightleftarrows \mathbb{C} \rightleftarrows 0$;

(vi) $Q_{(\{(1,1),(2,2)\},(1,2))}$ which consists of the zero quiver constructed as the direct sum of $0 \rightleftarrows 0 \rightleftarrows 0 \rightleftarrows \mathbb{C}^3$ and the zero quivers $0 \rightleftarrows \mathbb{C} \rightleftarrows 0 \rightleftarrows 0$ and $0 \rightleftarrows 0 \rightleftarrows \mathbb{C}^2 \rightleftarrows 0$.

Corollary 6.15. *Q^{hks} is a dense open subset of Q with complement of complex codimension at least* 2.

Proof. We observe using the complex equations that $\operatorname{rank}(\alpha_{i-1}\beta_{i-1} - \lambda_i^{\mathbb{C}} I) = \operatorname{rank}(\beta_i\alpha_i) < \min(\operatorname{rank}\alpha_i, \operatorname{rank}\beta_i)$, so if α_i or β_i has rank less than i then $\lambda_i^{\mathbb{C}}$ is an eigenvalue of $\alpha_{i-1}\beta_{i-1}$.

Now, as the map $(\alpha, \beta) \mapsto \beta\alpha$ is a surjection from $\operatorname{Hom}(\mathbb{C}^j, \mathbb{C}^{j+1}) \oplus \operatorname{Hom}(\mathbb{C}^{j+1}, \mathbb{C}^j)$ onto $\operatorname{Hom}(\mathbb{C}^j, \mathbb{C}^j)$ for all j we may vary the $\lambda_i^{\mathbb{C}}$ arbitrarily at each stage (without changing α_j, β_j for $j < i$), and still stay within $\mu_{\mathbb{C}}^{-1}(0)$. In particular the $H_{\mathbb{C}}$-invariant function $\Phi_i := \det(\alpha_{i-1}\beta_{i-1} - \lambda_i^{\mathbb{C}} I)$ cannot vanish identically on a non-empty open set in $\mu_{\mathbb{C}} = 0$. The complement of $\Phi_i^{-1}(0)$ is now open and dense in the locus where $\mu_{\mathbb{C}} = 0$ for each i.

This implies that Q^{hks} is dense in Q; the complement is a union of strata of even complex dimension, hence the codimension statement follows. $\square$

7. The structure of the strata

We shall now take a closer look at the structure of the strata $Q_{(S,\delta)}$ appearing in the stratification of the quiver space $Q = M /\!/\!/\!/ H$ with dimension vector $(n_1, \dots, n_r = n)$ given by Theorem 6.13. Recall that $Q_{(S,\delta)}$ is an open subset of a hyperkähler modification of a quotient of the form

$$M^{\text{hks}} /\!/\!/\!/ H_S = (M^{\text{hks}} /\!/\!/\!/ H) /\!/\!/\!/ T_S = Q^{\text{hks}} /\!/\!/\!/ T_S$$

for a different dimension vector $(m_1, \dots, m_r)$ and a subtorus T_S of $\tilde{T} = T^{r-1}$. Thus we shall first study the open stratum $Q^{\text{hks}} = Q_{(\varnothing,\varnothing)} = M^{\text{hks}} /\!/\!/\!/ H$ consisting of the hyperkähler stable quivers.

We first look at quivers

$$0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} \mathbb{C}^{n_1} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \mathbb{C}^{n_2} \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{r-2}}{\overset{\alpha_{r-2}}{\rightleftarrows}} \mathbb{C}^{n_{r-1}} \underset{\beta_{r-1}}{\overset{\alpha_{r-1}}{\rightleftarrows}} \mathbb{C}^{n_r} = \mathbb{C}^n$$

satisfying the complex moment map equations

$$\alpha_{i-1}\beta_{i-1} = \beta_i\alpha_i + \lambda_i^{\mathbb{C}} I_{n_i}$$

for H where $0 = n_0 \leqslant n_1 \leqslant \cdots \leqslant n_r = n$ and all the β_i are surjective. Lemma 5.30 shows that the $H_{\mathbb{C}}$-orbits of all such quivers are closed and hence represent points in Q^{hks} by Lemma 6.5.

As in the proof of Lemma 4.10 we may choose bases for the vector spaces $V_i = \mathbb{C}^{n_i}$ so that

$$\beta_i = (0_{n_i \times k_i} \mid I_{n_i \times n_i})$$

where $k_i = n_{i+1} - n_i$ is the dimension of the kernel of β_i. This amounts to using the action of $\tilde{H}_{\mathbb{C}} \times SL(n, \mathbb{C}) = \prod_{i=1}^r GL(n_i, \mathbb{C})$ to standardise the β_i, and we can replace $GL(n_i, \mathbb{C})$ with $SL(n_i, \mathbb{C})$ for each i such that $n_{i-1} < n_i$, so that if the dimension vector is strictly ordered then it amounts to using the action of $H_{\mathbb{C}} \times SL(n, \mathbb{C}) = \prod_{i=1}^r SL(n_i, \mathbb{C})$.

Let us now assume we are in the strictly ordered case. As we saw in Lemma 4.10, when the dimension vector is strictly ordered the remaining freedom in the group action is the commutator of the parabolic group P in $SL(n, \mathbb{C})$ associated to the flag of dimensions $(n_1, n_2, \dots, n_r = n)$ in $\mathbb{C}^n$. In the particular case when $n_i = i$ for all i (that is, all k_i equal 1), this freedom is exactly the maximal unipotent group N; that is, the commutator subgroup of the Borel group B.

Now let us investigate what $X = \alpha_{r-1}\beta_{r-1}$ tells us when the dimension vector is strictly ordered once the β_i have been standardised as above. With respect to bases chosen as above, the matrix of $\alpha_i\beta_i$ is

$$\begin{pmatrix} 0_{k_i \times k_i} & D_{k_i \times n_i} \\ 0_{n_i \times k_i} & -\lambda_i^{\mathbb{C}} I_{n_i} + \alpha_{i-1}\beta_{i-1} \end{pmatrix} \tag{7.1}$$

for some D.

Inductively it is now easy to show that $\alpha_i\beta_i$ has scalar blocks of size $k_j \times k_j$ ($j = i, i-1, \dots, 0$) down the diagonal, where the scalars (from top left going down) are $0, -\lambda_i^{\mathbb{C}}, -(\lambda_i^{\mathbb{C}} + \lambda_{i-1}^{\mathbb{C}}), \dots, -(\lambda_i^{\mathbb{C}} + \dots + \lambda_1^{\mathbb{C}})$.

In particular $X = \alpha_{r-1}\beta_{r-1}$ lies in the annihilator of the Lie algebra of the commutator $[P, P]$ of the parabolic determined by the integers k_j. Again, in the case $n_i = i$ we have that X lies in the Borel subalgebra $\mathfrak{b} = \mathfrak{n}^\circ$. Notice also that the diagonal entries of X are 0 (k_{r-1} times), $-\lambda_{r-1}^{\mathbb{C}}$ (k_{r-2} times), $\dots, -(\lambda_{r-1}^{\mathbb{C}} + \dots + \lambda_1^{\mathbb{C}})$ ($k_0 = n_1$ times).

Moreover any such X comes from a solution to our equations, because we have that X kills $\ker \beta_{r-1}$, and $\lambda_i^{\mathbb{C}} + \beta_i\alpha_i$ kills $\ker \beta_{i-1}$ for $i < r-1$.

Observe that for each i, knowledge of $\alpha_i\beta_i$ determines α_i (since β_i is surjective and standardised), hence determines $\beta_i\alpha_i$ (since β_i is standardised), and hence, together with knowledge of $\lambda_i^{\mathbb{C}}$, determines $\alpha_{i-1}\beta_{i-1}$ by the equations.

As we can read off the $\lambda_i^{\mathbb{C}}$ by looking at the diagonal entries of X, we see that knowledge of X determines all the α_i and hence the whole diagram.

In addition X is determined by its trace-free part, as its leading entry $X_{11} = 0$. So, in summary, we have shown, in the strictly ordered case, that if the β_i are surjective they may be standardised (modulo $H_{\mathbb{C}}$) by an element of $SL(n, \mathbb{C})$, unique up to an element of the commutator of the parabolic subgroup, and now the whole diagram is determined by X (or its trace-free part) in the annihilator of the Lie algebra of this commutator. Moreover any such X arises from such a diagram. We summarise our results as follows.

Proposition 7.2. *Consider a quiver diagram with strictly ordered dimension vector $(n_1, \dots, n_r = n)$. Then the set of solutions to the complex moment map equations for H with β_i surjective, modulo the action of $H_{\mathbb{C}} = \prod_{i=1}^{r-1} SL(n_i, \mathbb{C})$, may be identified with*

$$SL(n, \mathbb{C}) \times_{[P,P]} [\mathfrak{p}, \mathfrak{p}]^\circ$$

where P is the parabolic subgroup associated to the flag $(n_1, \dots, n_r = n)$, and $[\mathfrak{p}, \mathfrak{p}]^\circ$ is the annihilator of the Lie algebra of the commutator subgroup of P.

In the special (full flag) case where $n_i = i$ for all i, we obtain the space

$$SL(n, \mathbb{C}) \times_N \mathfrak{b}$$

where N is a maximal unipotent subgroup of $SL(n, \mathbb{C})$ and $\mathfrak{b} = \mathfrak{n}^\circ$ is a Borel subalgebra.

Remark 7.3. The space $SL(n, \mathbb{C}) \times_N \mathfrak{b}$ has also occurred in work of Bielawski [1, 2].

We also have:

Proposition 7.4. *Consider a quiver diagram with strictly ordered dimension vector* $(n_1, \dots, n_r = n)$*. Then the set of solutions to the complex moment map equations for* H *with* α_i *injective and* β_i *surjective, modulo the action of* $H_{\mathbb{C}} = \prod_{i=1}^{r-1} SL(n_i, \mathbb{C})$*, may be identified with*

$$SL(n, \mathbb{C}) \times_{[P,P]} [\mathfrak{p}, \mathfrak{p}]^\circ_*$$

where P *is the parabolic subgroup associated to the flag* $(n_1, \dots, n_r = n)$*, and* $[\mathfrak{p}, \mathfrak{p}]^\circ_*$ *is an open dense subset of* $[\mathfrak{p}, \mathfrak{p}]^\circ$*. Moreover* $[\mathfrak{p}, \mathfrak{p}]^\circ_*$ *is contained in the complement of the union over all parabolic subgroups* P' *strictly containing* P *of the annihilator* $[\mathfrak{p}', \mathfrak{p}']^\circ$ *of the Lie algebra of the commutator subgroup of* P'*.*

In the full flag case where $n_i = i$ *for all* i*, we obtain a space*

$$SL(n, \mathbb{C}) \times_N \mathfrak{b}_*$$

where N *is a maximal unipotent subgroup of* $SL(n, \mathbb{C})$ *and* $\mathfrak{b} = \mathfrak{n}^\circ$ *is a Borel subalgebra.*

Proof. The statement that X has to lie in the complement of $[\mathfrak{p}', \mathfrak{p}']^\circ$ follows by an induction, using (7.1) and the given form of β_i. □

How does the argument which gave us Propositions 7.2 and 7.4 need to be modified if the dimension vector $(n_1, \dots, n_r = n)$ is ordered but not strictly ordered? We may still choose bases for the vector spaces $V_i = \mathbb{C}^{n_i}$ so that

$$\beta_i = (0_{n_i \times k_i} \mid I_{n_i \times n_i})$$

where $k_i = n_{i+1} - n_i$ is the dimension of the kernel of β_i, but to do this we may need to use the action of a larger group than $H_{\mathbb{C}} \times SL(n, \mathbb{C}) = \prod_{i=1}^{r} SL(n_i, \mathbb{C})$. It suffices to use the action of $\tilde{H}_{\mathbb{C}} \times SL(n, \mathbb{C}) = \left(\prod_{i=1}^{r-1} GL(n_i, \mathbb{C})\right) \times SL(n_r, \mathbb{C})$, and then the remaining freedom is P itself, embedded in $\tilde{H}_{\mathbb{C}} \times SL(n, \mathbb{C})$ so that the projection of $g \in P$ in $GL(n_i, \mathbb{C})$ is the bottom right hand $n_i \times n_i$ block of g. Let

$$\tilde{T}_{\mathbb{C}} = \tilde{H}_{\mathbb{C}} / H_{\mathbb{C}} = \prod_{i=1}^{r-1} GL(n_i, \mathbb{C}) / SL(n_i, \mathbb{C}) = (\mathbb{C}^*)^{r-1}.$$

Once we have quotiented by the action of $H_{\mathbb{C}}$ we are using the residual action of $\tilde{T}_{\mathbb{C}} \times SL(n, \mathbb{C})$ to put the maps β_i into standard form, and the remaining freedom is the action of P embedded in $\tilde{T}_{\mathbb{C}} \times SL(n, \mathbb{C})$ via the inclusion in $SL(n, \mathbb{C})$ and the homomorphism

$$\chi = (\chi_1, \dots, \chi_{r-1}) \colon P \to \tilde{T}_{\mathbb{C}} = (\mathbb{C}^*)^{r-1}, \tag{7.5}$$

where $\chi_i \colon P \to \mathbb{C}^*$ is the character given by the determinant of the bottom right hand block of size $n_i \times n_i$. Note that the kernel of χ is the commutator subgroup $[P, P]$ of P, but that χ is not surjective if $k_i = 0$ (i.e. if $n_{i+1} = n_i$) for some i.

The complex moment map equations

$$\alpha_{i-1}\beta_{i-1} = \beta_i \alpha_i + \lambda_i^{\mathbb{C}} I_{n_i}$$

tell us that

$$\alpha_i = \begin{pmatrix} \alpha_i^{(1)} \\ \alpha_i^{(2)} \end{pmatrix},$$

where $\alpha_i^{(1)}$ is $k_i \times n_i$ and $\alpha_i^{(2)}$ is $n_i \times n_i$ and

$$\alpha_{i+1}^{(2)} = \begin{pmatrix} -\lambda_{i+1}^{\mathbb{C}} I_{k_i \times k_i} & \alpha_i^{(1)} \\ 0_{n_i \times k_i} & \alpha_i^{(2)} - \lambda_{i+1}^{\mathbb{C}} I_{n_i \times n_i} \end{pmatrix}.$$

Inductively it follows that $\alpha_i^{(2)}$ has scalar blocks of size $k_j \times k_j$ (for $j = r-1, \ldots, 0$) down the diagonal, where the scalars (from top left going down) are

$$-\lambda_i^{\mathbb{C}}, -(\lambda_i^{\mathbb{C}} + \lambda_{i-1}^{\mathbb{C}}), \ldots, -(\lambda_i^{\mathbb{C}} + \cdots + \lambda_1^{\mathbb{C}}).$$

Furthermore the quiver is determined by knowledge of $\lambda_1^{\mathbb{C}}, \ldots, \lambda_{r-1}^{\mathbb{C}}$ together with

$$X = \alpha_{r-1}\beta_{r-1} = \begin{pmatrix} 0_{k_{r-1} \times k_{r-1}} & \alpha_{r-1}^{(1)} \\ 0_{n_{r-1} \times k_{r-1}} & \alpha_{r-1}^{(2)} \end{pmatrix}$$

which is block triangular with scalar blocks of size $k_j \times k_j$ ($j = r-1, \ldots, 0$) down the diagonal, where the scalars are $0, -\lambda_{r-1}^{\mathbb{C}}, -(\lambda_{r-1}^{\mathbb{C}} + \lambda_{r-2}^{\mathbb{C}}), \ldots, -(\lambda_{r-1}^{\mathbb{C}} + \cdots + \lambda_1^{\mathbb{C}})$. Note that we can only recover from X those $\lambda_{r-1}^{\mathbb{C}} + \cdots + \lambda_i^{\mathbb{C}}$ for which $k_{i-1} > 0$; however from $\lambda_1^{\mathbb{C}}, \ldots, \lambda_{r-1}^{\mathbb{C}}$ (which we can regard as determining an element of the dual of the Lie algebra of $\tilde{T}_{\mathbb{C}}$) and the off-diagonal blocks of X (that is, the projection of X to $\mathfrak{p}^\circ$) we can recover the quiver with β_i in standardised form.

Remark 7.6. Now let S be an injective subrelation of $\leqslant$ on $\{1, \ldots, r-1\}$. Let T_S be the subtorus of $\tilde{T} = T^{r-1}$ as in Definition 5.26 and let $H_S = \phi^{-1}(T_S)$ be the corresponding subgroup of $\tilde{H}$ containing H. Then the complex moment map equations for H_S are given by the complex moment map equations

$$\alpha_{i-1}\beta_{i-1} = \beta_i\alpha_i + \lambda_i^{\mathbb{C}} I_{n_i}$$

for H together with the equations

$$\lambda_i^{\mathbb{C}} + \lambda_{i+1}^{\mathbb{C}} + \cdots + \lambda_j^{\mathbb{C}} = 0 \qquad \text{for } (i,j) \in S,$$

which say that $(\lambda_1^{\mathbb{C}}, \ldots, \lambda_{r-1}^{\mathbb{C}}) \in \mathrm{Lie}(\tilde{T}_{\mathbb{C}})^*$ lies in the annihilator of $\mathrm{Lie}(T_S)_{\mathbb{C}}$. Furthermore the residual action of $(H_S)_{\mathbb{C}}/H_{\mathbb{C}} = (T_S)_{\mathbb{C}}$ is given by its embedding as a subgroup of $\tilde{T}_{\mathbb{C}}$ and thus is a subgroup of $\tilde{T}_{\mathbb{C}} \times SL(n, \mathbb{C})$.

Putting all this together gives us the following extension of Proposition 7.4. It is a generalisation, since when the dimension vector is strictly ordered then the homomorphism $\chi\colon P \to \tilde{T}_{\mathbb{C}}$ defined at (7.5) above is surjective with kernel $[P,P]$, so that

$$(\tilde{T}_{\mathbb{C}} \times SL(n, \mathbb{C}))/P \cong SL(n, \mathbb{C})/[P,P].$$

Proposition 7.7. *Consider a quiver diagram with ordered dimension vector* $(n_1, \dots, n_r = n)$*. Then the set of solutions to the complex moment map equations for* H *with* α_i *injective and* β_i *surjective, modulo the action of* $H_{\mathbb{C}} = \prod_{i=1}^{r-1} SL(n_i, \mathbb{C})$*, may be identified with an open subset of the cotangent bundle to*

$$(\tilde{T}_{\mathbb{C}} \times SL(n, \mathbb{C}))/P,$$

or equivalently with

$$(\tilde{T}_{\mathbb{C}} \times SL(n, \mathbb{C})) \times_P (\mathrm{Lie}(\tilde{T}_{\mathbb{C}})^* \oplus \mathfrak{p}^\circ)^*$$

where P *is the parabolic subgroup associated to the flag* $(n_1, \dots, n_r = n)$*. Here*

$$\tilde{T}_{\mathbb{C}} = \tilde{H}_{\mathbb{C}}/H_{\mathbb{C}} = (\mathbb{C}^*)^{r-1}$$

with P *acting on* $SL(n, \mathbb{C})$ *by left multiplication and on* $\tilde{T}_{\mathbb{C}}$ *via the characters given by the determinants of the bottom right hand blocks of size* $n_i \times n_i$*, and* $(\mathrm{Lie}(\tilde{T}_{\mathbb{C}})^* \oplus \mathfrak{p}^\circ)^*$ *is an open dense subset of* $\mathrm{Lie}(\tilde{T}_{\mathbb{C}})^* \oplus \mathfrak{p}^\circ$*. Moreover the set of solutions to the complex moment map equations for* $H_S = \phi^{-1}(T_S)$ *(defined as in Definition 5.26) with* α_i *injective and* β_i *surjective, modulo the action of* $(H_S)_{\mathbb{C}}$*, may be identified with an open subset of the cotangent bundle to*

$$\tilde{T}_{\mathbb{C}} \times SL(n, \mathbb{C})/(T_S)_{\mathbb{C}} \times P$$

where $(T_S)_{\mathbb{C}} = (H_S)_{\mathbb{C}}/H_{\mathbb{C}}$ *is a subgroup of* $\tilde{T}_{\mathbb{C}}$ *and* $(T_S)_{\mathbb{C}} \times P$ *is embedded as a subgroup of* $\tilde{T}_{\mathbb{C}} \times SL(n, \mathbb{C})$ *via*

$$(t, g) \mapsto (t\chi(g), g)$$

with $\chi \colon P \to \tilde{T}_{\mathbb{C}}$ *as defined at* (7.5)*.*

Remark 7.8. An alternative argument notes that since the α_i are injective and the β_i are surjective then α_i and β_i are isomorphisms whenever $n_i = n_{i+1}$ and so we can contract the quiver as in Proposition 5.27 until we obtain a quiver of the same form but with strictly ordered dimension vector. For each such contraction the information lost is $\lambda_i^{\mathbb{C}} \in \mathbb{C}$ and the difference between the actions of $GL(V_{i+1})$ and $SL(V_{i+1})$ is $GL(V_{i+1})/SL(V_{i+1}) \cong \mathbb{C}^*$, so we can use this point of view to deduce Proposition 7.7 from Proposition 7.4.

Remark 7.9. In fact for certain values of $\lambda^{\mathbb{C}}$ we get restrictions on which parabolics can occur with a non-empty solution set in Proposition 7.7. More precisely, we observe that if $\lambda_{i+1}^{\mathbb{C}} = 0$ then the complex moment map equations imply α_{i+1} maps $\ker \beta_i$ into $\ker \beta_{i+1}$. If all β_j are surjective and α_j are injective, this means that $k_i \leqslant k_{i+1}$. In particular if all $\lambda_i^{\mathbb{C}}$ are zero, then the k_i form a *non-decreasing* partition of n. This is to be expected, as such partitions count the number of *unordered* partitions, that is, the strata of the nilpotent variety.

Remark 7.10. Now suppose that we are in the situation of Theorem 6.13. Recall that then the stratum $Q_{(S,\delta)}$ is the image of the hyperkähler embedding into $Q = M /\!\!/\!\!/ H$ defined in Proposition 6.9 with $S_1 = S$ and $S_2 = \varnothing$ of the hyperkähler modification $\hat{Q}_1^{\mathrm{hks}}$ of Q_1^{hks} as in (6.8).

Observe that $\hat{Q}_1^{\mathrm{hks}} = \hat{M}(\mathbf{m})^{\mathrm{hks}} /\!\!/\!\!/ H_S$, where

$$\hat{M}(\mathbf{m})^{\mathrm{hks}} = (M(\mathbf{m})^{\mathrm{hks}} \times (\mathbb{H} \setminus \{0\})^\ell) /\!\!/\!\!/ T^\ell. \tag{7.11}$$

Let $h = h_M - h_\ell$ be the hyperkähler moment map used in (7.11); this takes values in $\mathbb{R}^\ell \oplus \mathbb{C}^\ell$. We have that $\hat{M}(\mathbf{m})^{\mathrm{hks}} = (h^{\mathbb{C}})^{-1}(0) /\!\!/ T^\ell_{\mathbb{C}}$. Put $M_0 = (h_M^{\mathbb{C}})^{-1}((\mathbb{C} \setminus \{0\})^\ell)$. Now the ith component of $h_\ell^{\mathbb{C}}$ is just $a^{(i)} + \mathbf{j}b^{(i)} \mapsto a^{(i)}b^{(i)}$, so for $m \in M_0$, $h_M^{\mathbb{C}}(m) = h_\ell^{\mathbb{C}}((a^{(i)} + \mathbf{j}b^{(i)}))$ implies that each $a^{(i)}$ and $b^{(i)}$ is non-zero, and hence the $T^\ell_{\mathbb{C}}$ orbit through the point is closed. In particular, the holomorphic map $m \mapsto (m, h_M^{\mathbb{C}}(m) - \mathbf{j1})$, $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^\ell$, realises M_0 as an open subset of $h_{\mathbb{C}}^{-1}(0) /\!\!/ T^\ell_{\mathbb{C}} = \hat{M}(\mathbf{m})^{\mathrm{hks}}$. This map is equivariant for both $H_S^{\mathbb{C}}$ and $SL(n, \mathbb{C})$, and so descends to a holomorphic map on an open dense set of Q_1 to $\hat{Q}_1^{\mathrm{hks}}$. Exploiting the action of $SU(2)_{\mathrm{rotate}}$, we can cover $\hat{Q}_1^{\mathrm{hks}}$ by such open sets.

Now note that if $m_{j+1} = m_j$ for some j, then, since $n_{j+1} > n_j$ there exists some $i \leqslant j$ such that $(i, j) \in S$. This implies that the homomorphism

$$(T_S)_{\mathbb{C}} \times P \to \tilde{T}_{\mathbb{C}}$$

given by $(t, g) \mapsto t\chi(g)$ is surjective, and its kernel is isomorphic to the subgroup

$$P_S = \{g \in P : \chi(g) \in (T_S)_{\mathbb{C}}\}$$

of P containing $[P, P]$. Thus

$$\tilde{T}_{\mathbb{C}} \times SL(n, \mathbb{C})/(T_S)_{\mathbb{C}} \times P$$

can be identified with $SL(n, \mathbb{C})/P_S$, and its cotangent bundle can be identified with

$$SL(n, \mathbb{C}) \times_{P_S} \mathfrak{p}_S^\circ$$

where $\mathfrak{p}^\circ$ is the annihilator in the dual of the Lie algebra of $SL(n, \mathbb{C})$ of the Lie algebra $\mathfrak{p}_S$ of P_S.

Thus we have

Theorem 7.12. *In the situation of Theorem 6.13 each stratum $Q_{(S,\delta)}$ of Q is a union of open subsets, one for each element of $SU(2)_{\mathrm{rotate}}$, each of which can be identified with*

$$SL(n, \mathbb{C}) \times_{P_S} (\mathfrak{p}_S)^\circ_*.$$

Here $(\mathfrak{p}_S)^\circ_*$ is an open subset of the annihilator $(\mathfrak{p}_S)^\circ$ in $\mathrm{Lie}(SL(n, \mathbb{C}))^*$ *of the Lie algebra $\mathfrak{p}_S$ of a subgroup P_S of the standard parabolic subgroup P of $SL(n, \mathbb{C})$ associated to the flag $\mathbb{C}^{m_1} \leqslant \cdots \leqslant \mathbb{C}^{m_r} = \mathbb{C}^n$ where*

$$m_k = k - d_k,$$

d_k the kth component of $\mathbf{d} = \sum_{(i,j)\in S} \delta(i)e_{ij}$. More precisely, $P_S = \{g \in P : \chi(g) \in (T_S)_\mathbb{C}\}$ where $(T_S)_\mathbb{C} = (H_S)_\mathbb{C}/H_\mathbb{C}$ is the subgroup of $\tilde{T}_\mathbb{C} = \tilde{H}_\mathbb{C}/H_\mathbb{C}$ defined as in Definition 5.26 and $\chi\colon P \to \tilde{T}_\mathbb{C}$ is defined at (7.5)*.*

Corollary 7.13. *In the case of a full flag, when $r = n$ and $n_j = j$ for $j = 0, \dots, n$, this description applies to each stratum $Q_{(S,\delta)}$ of the stratification described in Theorem 6.13 of the universal hyperkähler implosion $Q = M /\!\!/\!\!/ H$ for $K = SU(n)$.*

Remark 7.14. $SL(n,\mathbb{C}) \times_{P_S} \mathfrak{p}_S^\circ$ can be regarded as the complex-symplectic quotient of $T^*K_\mathbb{C} = K_\mathbb{C} \times \mathfrak{k}_\mathbb{C}$ by P_S, where $K = SU(n)$ and $[P,P] \leqslant P_S \leqslant P$. We may compare this, as in §2, with symplectic implosion, where the universal implosion $(T^*K)_{\mathrm{impl}} = K_\mathbb{C} /\!\!/ N$ is stratified by the ordinary quotients of $K_\mathbb{C}$ by commutators of parabolics.

Remark 7.15. We have noted that for general compact K, the space $K_\mathbb{C} \times_{[P,P]} [\mathfrak{p},\mathfrak{p}]^\circ$ may be viewed as the cotangent bundle of $K_\mathbb{C}/[P,P]$. Now, from [9] the latter quotient is just a Kähler stratum of the symplectic implosion $K_\mathbb{C} /\!\!/ N$ (cf. Remark 7.14). A theorem of Feix [8], now shows that there is a hyperkähler metric on some open neighbourhood of the zero section in the cotangent bundle $K_\mathbb{C} \times_{[P,P]} [\mathfrak{p},\mathfrak{p}]^\circ$. Proposition 7.2 gives us a hyperkähler structure on the full set $K_\mathbb{C} \times_{[P,P]} [\mathfrak{p},\mathfrak{p}]^\circ$ for $K = SU(n)$

Remark 7.16. Note that in the full flag case the homomorphism $\chi\colon B \to \tilde{T}_\mathbb{C}$ defined at (7.5) is surjective with kernel $N = [B,B]$ and thus allows us to identify $\tilde{T}_\mathbb{C}$ naturally with the maximal torus $T_\mathbb{C}$ of $K_\mathbb{C}$.

We would also like to relate the quiver space $Q = M /\!\!/\!\!/ H$ in the full flag case to the non-reductive GIT quotient $(SL(n,\mathbb{C}) \times \mathfrak{b}) /\!\!/ N$, which as discussed in §2 could be interpreted as a complex-symplectic quotient in the GIT sense of the cotangent bundle $T^*\,SL(n,\mathbb{C})$ by the maximal unipotent N.

Lemma 7.17. *When $r = n$ and $n_j = j$ for $j = 0, \dots, n$ the complement in the variety defined by the complex moment map equations $\mu_\mathbb{C} = 0$ of the locus of full flag quivers with all α_i injective and β_i surjective has complex codimension at least* 2*.*

Proof. As in the proof of Corollary 6.15, we observe that if some α_i or β_i is of less than maximal rank, then the $H_\mathbb{C}$-invariant function Φ_i given by $\det(\alpha_{i-1}\beta_{i-1} - \lambda_i^\mathbb{C} I)$ is zero; moreover we may vary the $\lambda_i^\mathbb{C}$ arbitrarily at each stage (without changing α_j, β_j for $j < i$), and still stay within $\mu_\mathbb{C}^{-1}(0)$ since the map $(\alpha,\beta) \mapsto \beta\alpha$ is a surjection from $\mathrm{Hom}(\mathbb{C}^j, \mathbb{C}^{j+1}) \oplus \mathrm{Hom}(\mathbb{C}^{j+1}, \mathbb{C}^j)$ onto $\mathrm{Hom}(\mathbb{C}^j, \mathbb{C}^j)$ for all j. In particular the the zero locus of Φ_i is a variety of codimension one. Furthermore, if this occurs for two indices i and j, we can see in the

same way that both Φ_i and Φ_j vanish, and this is a codimension two condition because Φ_j does not vanish identically on a non-empty open set in the zero locus of Φ_i.

Similarly, if for some i the rank of α_i or β_i is less than $i-1$, we deduce that Φ_i vanishes to order at least two, so we are in codimension two or higher.

So we just have to consider the situation where there is only one index i where α_i or β_i are of less than full rank, and for this index $(\operatorname{rank}\alpha_i, \operatorname{rank}\beta_i) = (i, i-1), (i-1, i)$ or $(i-1, i-1)$.

If β_i is of maximal rank, we can put it in the standard form of Proposition 7.2, and now α_i is a $(i+1) \times i$ matrix such that the bottom $i \times i$ block is $\alpha_{i-1}\beta_{i-1} - \lambda_i^{\mathbb{C}} I_{i\times i}$. The vanishing of Φ_i just says that the associated minor determinant is zero. The condition that α_i is of rank $i-1$ means also that the other $i \times i$ minors must vanish, so we get a subvariety of codimension at least two in $\mu_{\mathbb{C}}^{-1}(0)$. The case $(i-1, i)$ follows by dualising, and the third case $(i-1, i-1)$ proceeds by a similar calculation choosing a standard form for the rank $i-1$ map β_i. □

Theorem 7.18. *The algebra of invariants $\mathcal{O}(SL(n,\mathbb{C}) \times \mathfrak{b})^N$ is finitely generated, and the hyperkähler quotient $Q = M/\!/\!/H$ of the space M of full flag quivers by H can be identified with the non-reductive GIT quotient*

$$(SL(n,\mathbb{C}) \times \mathfrak{b})/\!/N.$$

Proof. The set $\mu_{\mathbb{C}}^{-1}(0)^{\mathrm{surj}}$ of full flag quivers satisfying the complex equations and with all β_i surjective is, by Lemma 7.17, an open set in $\mu_{\mathbb{C}}^{-1}(0)$ whose complement is of complex codimension at least two. Moreover, the proof of Proposition 7.2 shows that $\mu_{\mathbb{C}}^{-1}(0)^{\mathrm{surj}}$ may be identified with $(SL(n,\mathbb{C}) \times H_{\mathbb{C}}) \times_N \mathfrak{b}$.

Therefore the coordinate algebra $\mathcal{O}(\mu_{\mathbb{C}}^{-1}(0))^{H_{\mathbb{C}}}$ is isomorphic to

$$\mathcal{O}((SL(n,\mathbb{C}) \times H_{\mathbb{C}}) \times_N \mathfrak{b})^{H_{\mathbb{C}}},$$

and hence to $\mathcal{O}(SL(n,\mathbb{C}) \times \mathfrak{b})^N$. As $\mu_{\mathbb{C}}^{-1}(0)$ is an affine variety and $H_{\mathbb{C}}$ is reductive, this algebra is finitely generated. Moreover

$$Q = \mu_{\mathbb{C}}^{-1}(0)/\!/H_{\mathbb{C}} = \operatorname{Spec}\mathcal{O}(\mu_{\mathbb{C}}^{-1}(0))^{H_{\mathbb{C}}} = \operatorname{Spec}\mathcal{O}(SL(n,\mathbb{C}) \times \mathfrak{b})^N,$$

and the last space is by definition the non-reductive quotient $(SL(n,\mathbb{C}) \times \mathfrak{b})/\!/N$ (cf. the discussion in §2), so the result follows. □

Together with the results of §3, this means that the complex-symplectic quotients of the implosion will give the Kostant varieties V_χ.

Remark 7.19. The symplectic implosion $(T^*K)_{\mathrm{impl}}$ has an action of $\mathbb{R}^*$ induced from multiplication in the fibres of T^*K. Similarly, we have a $\mathbb{C}^*$ action on $T^*K_{\mathbb{C}}$ given by $(g,\xi) \mapsto (g, \tau\xi)$. This action commutes with the right $K_{\mathbb{C}}$ action (2.1), and hence, for each P_S, preserves the property of being in the zero level set for the complex-symplectic action

of P_S and induces an action on the subsets $K_{\mathbb{C}} \times_{P_S} \mathfrak{p}_S^{\circ}$. Identifying these with the cotangent bundles of $K_{\mathbb{C}}/P_S$, this is just scaling in the fibre.

In fact, this extends to an action on the full implosion space Q, given by just scaling the β_i. Notice that this will also scale $X = \alpha_{n-1}\beta_{n-1}$, so will induce the scaling in the fibre of the cotangent bundles above. We can also view it as the action on $(SL(n, \mathbb{C}) \times \mathfrak{b}) /\!\!/ N$ induced by scaling the $\mathfrak{b}$ factor.

Theorem 7.20. *The fixed points of the $\mathbb{C}^*$ action on the universal hyperkähler implosion Q given by scaling the β_i are represented by the quivers with $\beta_i = 0$ for each i. The fixed point set may therefore be identified, by the discussion in §4, with the universal symplectic implosion for $K = SU(n)$.*

Proof. If (α, β) is fixed by the action, then for all $\tau \in \mathbb{C}^*$ there exists $g_\tau \in SL$ with $g_\tau.(\alpha, \beta) = (\alpha, \tau\beta)$. Letting $\tau \to 0$, we see that $(\alpha, 0)$ is in the closure of the SL-orbit of (α, β), and hence in the same orbit by our polystability condition. It now follows that $\beta = 0$. □

Proposition 7.21. *If the universal hyperkähler implosion Q for $SU(n)$ is smooth, then so is the universal symplectic implosion for $SU(n)$, and hence $n \leqslant 2$.*

Proof. We consider the action of the maximal compact subgroup S^1 of $\mathbb{C}^*$. If the hyperkähler implosion were smooth, then the fixed point set of this circle action would also be smooth. By general properties of reductive group actions, this set is also the fixed point set of the $\mathbb{C}^*$ action, which from above is just the universal symplectic implosion. The result now follows from [9, §6] which tells us that the universal symplectic implosion for K is smooth if and only if the commutator $[K, K]$ is a product of copies of $SU(2)$. □

Remark 7.22. We shall see in Example 8.5 that the universal hyperkähler implosion *is* smooth if $K = SU(2)$.

8. Geometry of the strata and torus reductions

Let us now further investigate the geometry of the strata described in Theorems 6.13 and 7.12 and consider some examples. We particularly focus on the stratification of the universal hyperkähler implosion Q of $K = SU(n)$, where the original quiver is a full flag, that is, $r = n$ and $n_j = j$ for each j. Of course, the strata will involve quivers that are not necessarily of full flag type.

For the purposes of considering torus reductions of the implosion, we need to focus on the $\tilde{T}_{\mathbb{C}}$-polystable locus; that is, the locus where the action of $\tilde{H}_{\mathbb{C}} = \prod_{i=1}^{r-1} GL(n_i, \mathbb{C})$ on the quivers is polystable. (Recall from Remark 7.16 that in the full flag case we have a canonical identification of $\tilde{T}_{\mathbb{C}}$ with the maximal torus $T_{\mathbb{C}}$ of $K_{\mathbb{C}}$). Using the results of §5 and §7 we can relate these loci to open subsets of the cotangent

bundles $SL(n,\mathbb{C}) \times_{P_S} \mathfrak{p}_S^\circ$ of $SL(n,\mathbb{C})/P_S$ where $[P,P] \leqslant P_S \leqslant P$ for suitable parabolics P.

In particular, we shall first take the parabolic P to be the Borel subgroup B with $P_S = [B,B] = N$, and look at the subset $SL(n,\mathbb{C}) \times_N \mathfrak{b}$ in the implosion, which as we have seen corresponds to full flag quivers with all β_i surjective.

In order to analyse the N action, let us write

$$\mathfrak{k}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{k}_\alpha \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{k}_{-\alpha},$$

where $\mathfrak{k}_\alpha$ are the root spaces and Δ_+ the set of positive roots. So $[\mathfrak{k}_\alpha, \mathfrak{k}_\beta] \subset \mathfrak{k}_{\alpha+\beta}$, taking $\mathfrak{k}_0$ to be $\mathfrak{t}_\mathbb{C}$ and $\mathfrak{k}_\gamma$ to be zero if γ is not a root. Recall also that $\mathfrak{k}_\alpha$ and $\mathfrak{k}_\beta$ are Killing-orthogonal if $\alpha + \beta \neq 0$. We are using the pairing $(A,B) \mapsto \mathrm{tr}(AB)$ to identify $\mathfrak{k}$ and $\mathfrak{k}_\mathbb{C}$ with their duals.

We can take

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{k}_\alpha$$

and

$$\mathfrak{n}^\circ = \mathfrak{b} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{k}_\alpha\,.$$

Let $X_{(\alpha)}$ denote the component of X in $\mathfrak{k}_\alpha$. So if $n \in \mathfrak{n}$ and $X \in \mathfrak{n}^\circ = \mathfrak{b}$ then $[n,X]_{(\alpha)}$ is $[n_{(\alpha)}, X_{(0)}]$ plus terms $[n_{(\beta)}, X_{(\gamma)}]$ where $\beta + \gamma = \alpha$ and $\beta, \gamma > 0$. Moreover $[n_{(\alpha)}, X_{(0)}]$ is just $-\alpha(X_{(0)}) n_{(\alpha)}$.

The adjoint action of $\exp(n) \in N$ on $X \in \mathfrak{k}_\alpha$ is

$$X \mapsto X + [n,X] + \text{terms in higher iterated brackets.}$$

So

$$(\exp(n)X)_{(0)} = X_{(0)}$$

and

$$(\exp(n)X)_{(\alpha)} = X_{(\alpha)} - \alpha(X_{(0)}) n_{(\alpha)} + \cdots$$

where $\cdots$ denotes terms in $n_{(\beta)}, X_{(\gamma)}$ with β (respectively γ) ranging over positive roots less than α (respectively 0 or positive roots less than α).

This means that, provided α does not vanish on the Cartan component of X, we may work up inductively through the root spaces, starting with the lowest, finding $n_{(\alpha)}$ such that $(\exp(n)X)_{(\alpha)} = 0$. Moreover these $n_{(\alpha)}$ are uniquely determined.

So if $X_{(0)}$ lies in the complement of the union of the zero loci of the roots then the N-orbit through X contains a unique element in the chosen Cartan algebra $\mathfrak{t}_\mathbb{C}$. (See Example 8.7 for a concrete example in the $SU(3)$ case).

We see that if the eigenvalues of $X \in \mathfrak{b}$ are all distinct, then the corresponding part of $SL(n,\mathbb{C}) \times_N \mathfrak{b}$ may be identified with $SL(n,\mathbb{C}) \times \mathfrak{t}_\mathbb{C}^{\mathrm{reg}}$, where $\mathfrak{t}_\mathbb{C}^{\mathrm{reg}}$ is the complement of the union of the zero loci for the roots in $\mathfrak{t}_\mathbb{C}$; that is, the set of diagonal matrices in $\mathfrak{sl}(n,\mathbb{C})$ with

distinct entries. Note that in the setup of Proposition 7.2, this amounts to standardising the α_i so that the only non-zero entries are in position $(j+1, j)$ for $j = 1, \dots, i$. The fact that the eigenvalues of X are distinct now implies that all α_i are injective.

Proposition 8.1. *The implosion Q contains $SL(n, \mathbb{C}) \times \mathfrak{t}_{\mathbb{C}}^{\mathrm{reg}}$ as an open dense subset.*

Remark 8.2. Notice that (for $K = SU(n)$) $K_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}^{\mathrm{reg}}$ is in this sense a complex-symplectic analogue of the product of K with the interior of the Weyl chamber, which is the open stratum in the symplectic implosion of T^*K. However in our case $SL(n, \mathbb{C}) \times \mathfrak{t}_{\mathbb{C}}^{\mathrm{reg}}$ is a proper subset of $SL(n, \mathbb{C}) \times_N \mathfrak{b}_*$, and hence of the open hyperkähler stratum, because configurations with equal eigenvalues may occur in Q^{hks}.

Let us now look at strata corresponding to more general parabolics. Our discussion of the diagonal entries of X in Proposition 7.2 shows that the eigenvalue κ_i of the trace-free part of X occurs at least k_{i-1} times ($i = 1, \dots, r$ with the convention that $\kappa_r = 0$). We say 'at least' because if $\lambda_i^{\mathbb{C}} + \lambda_{i+1}^{\mathbb{C}} + \dots \lambda_j^{\mathbb{C}}$ is zero for $i \leqslant j$ then κ_i and κ_{j+1} will be equal.

Hence, given a collection of eigenvalues κ_j for X, the maximum value of k_{i-1} compatible with this collection is the multiplicity of the corresponding κ_i.

Example 8.3. In particular if the eigenvalues of X are all distinct, then the k_i are all 1 and we are in the full flag case where $n_i = i$ for all i. This means we are in $SL(n, \mathbb{C}) \times \mathfrak{t}_{\mathbb{C}}^{\mathrm{reg}}$, which is open and dense in $SL(n, \mathbb{C}) \times_N \mathfrak{b}$.

Performing hyperkähler reduction by the maximal torus T is equivalent to fixing the $\mathfrak{t}_{\mathbb{C}}$-component of $\mathfrak{b}$, i.e. fixing $X_{(0)}$, and quotienting by $T_{\mathbb{C}}$. If we reduce at a level in $\mathfrak{t}_{\mathbb{C}}^{\mathrm{reg}}$ then the above discussion shows the resulting space is the semisimple orbit $SL(n, \mathbb{C})/T_{\mathbb{C}}$. $\diamondsuit$

Now let us consider reduction at a level $X_{(0)}$ where a root vanishes. If $\alpha(X_{(0)}) \neq 0$, then as above we can set $(\exp(n)X)_{(\alpha)}$ to zero. That is, we can use the N action to move X into the subalgebra $\mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}_1$, where

$$\mathfrak{n}_1 = \bigoplus_{\substack{\alpha \in \Delta_+ \\ \alpha(X_{(0)}) = 0}} \mathfrak{k}_\alpha .$$

We have left a residual action of N_1, the unipotent group with Lie algebra $\mathfrak{n}_1$.

Example 8.4. In particular, suppose X has distinct eigenvalues $\sigma_1, \dots, \sigma_s$ with multiplicities $m_1, \dots, m_s$ and let us take the maximum possible k_i compatible with this: that is, we take $r = s$ and $k_i = m_{i+1}$, $i = 0, \dots, r-1$. Let P denote the associated parabolic, whose commutator contains the maximal unipotent N. As we are concerned with

the quotients of the strata by the full torus $\tilde{T}_{\mathbb{C}}$ we shall assume in the following discussion that $P_S = [P, P]$; this will not result in any loss of generality as regards the quotient, as P_S is an extension of $[P, P]$ by a subtorus of $\tilde{T}_{\mathbb{C}}$.

We can now use part of the N action to kill the entries of X in positions (i, j) where $X_{ii} \neq X_{jj}$. But our choice of k_i means the other non-diagonal elements of X are zero, as X has to be in the annihilator $[\mathfrak{p}, \mathfrak{p}]^\circ$. The remaining part of $[P, P]$ that we have not used is just $\prod_{i=1}^{r-1} SL(k_i, \mathbb{C})$.

So we obtain $SL(n, \mathbb{C}) \times \mathfrak{t}_{\mathbb{C}}^{(\sigma,m)} / \prod_{i=1}^{r-1} SL(k_i, \mathbb{C})$, where $\mathfrak{t}_{\mathbb{C}}^{(\sigma,m)}$ denotes the subset of $\mathfrak{t}_{\mathbb{C}}$ satisfying the above equalities of eigenvalues. Hyperkähler reduction now fixes the diagonal entries and quotients by $T_{\mathbb{C}}$, hence we obtain the general semisimple orbit

$$SL(n, \mathbb{C})/S\Big(\prod_{i=1}^{r-1} GL(k_i, \mathbb{C})\Big)$$

Thus we obtain the semisimple orbits (the closed stratum of the Kostant variety) by choosing the stratum where the k_i are the maximum possible given the eigenvalues. The case of distinct eigenvalues, where all k_i must be 1, gives the regular semisimple orbit as in the preceding example. $\diamondsuit$

In general we have that each eigenvalue multiplicity m_i, $i = 1, \ldots, s$, is a sum of k_j, say $k_{i_1} + \cdots + k_{i_p}$. We can use part of the $[P, P]$ action to reduce X to block-diagonal form where we have one block for each distinct eigenvalue σ_i. Moreover each block is upper triangular with diagonal entries all equal to the eigenvalue σ_i.

The remaining freedom in $[P, P]$ is now also block-diagonal: each block is the commutator of a parabolic P_{σ_i} in $GL(m_i, \mathbb{C})$. Let us write P_{σ_i} as $U_{\sigma_i} L_{\sigma_i}$ where U_{σ_i} is the unipotent radical of P_{σ_i} and L_{σ_i} is the corresponding Levi subgroup.

The condition that X lies in $[\mathfrak{p}, \mathfrak{p}]^\circ$ means that, for each block, (transposing and dualising), the non-scalar part of that block of X actually lies in $\mathfrak{u}_{\sigma_i}$. The scalar part of the block is of course just $\sigma_i I_{m_i \times m_i}$.

The previous example is the case when $r = s$ and $k_i = m_{i+1}$, so we have one k_i for each distinct eigenvalue. Now $P_{\sigma_i} = GL(m_i, \mathbb{C})$ so the unipotent U_{σ_i} is scalar and hence X is scalar on each block. Moreover the remaining freedom in $[P, P]$ is just the product of the Levi subgroups $L_{\sigma_i} = SL(k_{i-1}, \mathbb{C})$, in agreement with the results of that example.

For some concrete low-dimensional examples consider the following.

Example 8.5. Let us take $K = SU(2)$.

(i) The quivers we have to consider are now of the form

$$\mathbb{C} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{C}^2$$

so we have $M = \mathbb{H}^2 = \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \oplus \mathrm{Hom}(\mathbb{C}^2, \mathbb{C})$. In the terminology of §5 the group $\tilde{H}$ is $U(1)$ but its commutator H and the associated complex group SL are the trivial groups $SU(1)$ and $SL(1, \mathbb{C})$. There is therefore no moment map equation and no stability condition for these groups, and the implosion Q is just $\mathbb{H}^2$. We have a hyperkähler action of the torus $T = U(1)$.

We can decompose the implosion into 4 subsets. The smallest one $\mathcal{S}_{\mathrm{bottom}}$ is when α, β are both zero, so $\mathbb{C} = \ker \alpha \oplus \operatorname{im} \beta$. At the other extreme, we let $\mathcal{S}_0$ be the set of quivers when α is injective and β surjective, so again this direct sum condition holds. This subset is $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$. We also have a subset $\mathcal{S}_1$ with β surjective but α not injective (so zero), and a subset $\mathcal{S}_2$ with α injective and β not surjective (so zero). Both these subsets are isomorphic to $\mathbb{C}^2 \setminus \{0\}$.

We observe that $\mathcal{S}_0$ and $\mathcal{S}_{\mathrm{bottom}}$ are sets of the form described in Proposition 7.4, corresponding to choosing the parabolic P to be the Borel B and the full group $SL(2, \mathbb{C})$, respectively. To see this, we can take the unipotent subgroup N of $K_{\mathbb{C}}$ to be the group of upper triangular matrices

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{C} \right\}.$$

We must consider $SL(2, \mathbb{C}) \times \mathfrak{n}^\circ$, where

$$\mathfrak{n}^\circ = \mathfrak{b} = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

Writing an element of $SL(2, \mathbb{C})$ as $\left(\begin{smallmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{smallmatrix}\right)$, the N action is

$$\begin{aligned} q_{12} &\mapsto q_{12} - t q_{11} \\ q_{22} &\mapsto q_{22} - t q_{21} \\ b &\mapsto b - 2at \end{aligned}$$

while q_{11}, q_{21}, a are invariant. Also $q_{11} q_{22} - q_{12} q_{21} = 1$.

According to the discussion in §7, $\mathcal{S}_0$ will be just the open dense set $SL(2, \mathbb{C}) \times_N \mathfrak{b}_*$ in $SL(2, \mathbb{C}) \times_N \mathfrak{b}$, where $X \in \mathfrak{b}$ is non-zero (that is, a, b are not both zero).

If a is non-zero then taking the N quotient is equivalent to setting $b = 0$. We obtain a set $SL(2, \mathbb{C}) \times (\mathbb{C} \setminus \{0\})$, which may be identified with $(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C} \times (\mathbb{C} \setminus \{0\})$ (viewing the $\mathbb{C}^2 \setminus \{0\}$ factor as the first column of the matrix in $SL(2, \mathbb{C})$).

If a is zero then the N action on the $\mathfrak{b}$ factor is trivial, and we obtain $(SL(2, \mathbb{C})/N) \times \{X \in \mathfrak{b} : a = 0,\ b \neq 0\}$, which is just $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$. Identification of $(SL(2, \mathbb{C})/N$ with $\mathbb{C}^2 \setminus \{0\}$ follows,

for example, from the Iwasawa decomposition—this space is of course the open stratum of the symplectic implosion for $SU(2)$).

The whole of $\mathcal{S}_0$, then, may be viewed as $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$ in accordance with the quiver picture. On the other hand $\mathcal{S}_{\text{bottom}}$ has $X = 0$ so is just $SL(2,\mathbb{C})/SL(2,\mathbb{C})$; that is, a point, again agreeing with the quiver description above.

Note that if instead we took the whole of $SL(2,\mathbb{C}) \times_N \mathfrak{b}$, rather than requiring $X \neq 0$, then we would obtain $(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}^2$. This of course corresponds to the union of the two sets $\mathcal{S}_0$ and $\mathcal{S}_1$, that is, quivers with β surjective, in accordance with Proposition 7.2.

In terms of the hyperkähler stratification in §6, the open stratum is the union $\mathbb{H}^2 \setminus \{0\}$ of $\mathcal{S}_0, \mathcal{S}_1$ and $\mathcal{S}_2$, which can also be viewed as the $SU(2)_{\text{rotate}}$ sweep of $\mathcal{S}_0$. The closed stratum is just $\mathcal{S}_{\text{bottom}}$, which is the origin. In the language of Proposition 6.9, the open stratum corresponds to taking S empty, so there is no δ, while the closed stratum corresponds to taking $S = \{(1,1)\}$ with $p = h = k = d_1 = 1$ and $0 = m_0 = m_1 < m_2 = 2$.

From the point of view of reductions by the torus $U(1)$ the relevant sets are $\mathcal{S}_{\text{bottom}}$ and $\mathcal{S}_0$ because these give the closed orbits for the complexified action on $\mathbb{H}^2$, in agreement with Proposition 5.20. The union of these strata can be viewed as the set of pairs (z, w) in $\mathbb{C}^2 \times \mathbb{C}^2$ such that z, w are both zero or both non-zero.

The hyperkähler quotient by $U(1)$ gives us the hyperkähler structure on Kostant varieties of $SL(2,\mathbb{C})$ (i.e. Eguchi-Hanson or the nilpotent variety) as explained below.

(ii) It is also instructive, in the light of Theorem 7.18, to consider the GIT quotient $(SL(2,\mathbb{C}) \times \mathfrak{b})/\!\!/N$. Using the variables above, the invariant polynomials are generated by q_{11}, q_{21}, a and $y_1 = 2q_{22}a - q_{21}b, y_2 = 2q_{12}a - q_{11}b$. We have the relation

$$q_{11}y_1 - q_{21}y_2 = 2a, \tag{8.6}$$

so $(SL(2,\mathbb{C}) \times \mathfrak{b})/\!\!/N$ is the affine hypersurface in $\mathbb{C}^5$ with equation (8.6). Now projection onto $(q_{11}, q_{21}, y_1, y_2)$ gives an isomorphism with $\mathbb{C}^4 = \mathbb{H}^2$.

The $\mathbb{C}^*$-action of Remark 7.19 is just scaling of a, b, hence of a, y_1, y_2. Its fixed-point set on $\mathbb{C}^4 = \mathbb{H}^2$ is just given by $y = 0$ and hence is a copy of $\mathbb{C}^2$, the symplectic implosion of $T^* SU(2)$.

The $T_{\mathbb{C}}$ action is $q_{i1} \mapsto s^{-1}q_{i1}$, $q_{i2} \mapsto sq_{i2}$, $b \mapsto s^2 b$ while a is invariant. So $y_i \mapsto sy_i$, and equation (8.6) is preserved.

Under the above identification with $\mathbb{C}^4$, the coordinates q_{11}, q_{21} scale by s^{-1} and y_1, y_2 by s. Note that the non-closed orbits for the $T_{\mathbb{C}}$ action are therefore those lying in $q_{11} = q_{12} = 0$ and $y_1 = y_2 = 0$, apart from the origin which is a closed point orbit.

Now, it is well-known that the hyperkähler reduction of flat $\mathbb{H}^2$ by $U(1)$ at the generic level gives the Eguchi-Hanson metric on the semisimple orbit of $SL(2,\mathbb{C})$. In terms of our variables, making the reduction is equivalent to fixing the value of a (i.e. the value of the complex-symplectic moment map for $T_{\mathbb{C}}$), and then quotienting by $T_{\mathbb{C}}$. Taking as invariant polynomials on $\mathbb{C}^4$ the expressions $W = q_{11}y_1, Y = q_{11}y_2, Z = q_{21}y_1$ and $q_{21}y_2$ (which equals $W - 2a$), we obtain the affine surface $W(W-2a) = YZ$, which is one of the complex structures for Eguchi-Hanson. If we reduce $\mathbb{H}^2$ by $U(1)$ at level 0, of course, we get the nilpotent variety $W^2 = YZ$ of $SL(2,\mathbb{C})$ with singularity at the origin. $\diamondsuit$

Example 8.7. Let us take $K = SU(3)$. From the quiver picture, we see the implosion is a hyperkähler quotient of $\mathbb{H}^8$ by $SU(2)$; in fact it may be realised as the Swann bundle (with origin adjoined) of the quaternionic Kähler manifold $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^8)$ of oriented 4-planes in $\mathbb{R}^8$. Note that this space is not smooth but has a conical singularity at the origin.

The N action on $\mathfrak{b} = \mathfrak{n}^\circ$ is:

$$\begin{pmatrix} 1 & r & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \longmapsto \begin{pmatrix} a & b + r(d-a) & \binom{c+rt(a-d)-tb}{+re+s(f-a)} \\ 0 & d & e + t(f-d) \\ 0 & 0 & f \end{pmatrix}$$

where $a + d + f = 0$, of course.

Let us first take a, d, f distinct, so we must be in the open stratum. As we are on the complement of the union of zero loci of the roots $a - d, d - f, f - a$, taking the quotient by N is equivalent to setting $b = c = e = 0$. So we obtain, as in Proposition 8.1, a set which can be identified with $SL(3,\mathbb{C}) \times \mathfrak{t}_{\mathbb{C}}^{\mathrm{reg}}$.

The hyperkähler reduction at such a level by the action of the maximal torus T will be just the complex-symplectic quotient by the complex torus $T_{\mathbb{C}}$. This will be obtained by fixing the value of (a, d, f) and then factoring out by $T_{\mathbb{C}}$, so we get $SL(3,\mathbb{C})/T_{\mathbb{C}}$ which is a semisimple orbit for $SL(3,\mathbb{C})$.

Now consider the case of eigenvalues $(a, a, -2a)$ with $a \neq 0$, i.e. when $a = d \neq f$. We can obtain this configuration by reducing at level $(\lambda_1^{\mathbb{C}}, \lambda_2^{\mathbb{C}}) = (3a, 0)$ in the stratum $Q_{(\{2,2\},1)}$, which has $k_0 = 1, k_1 = 0, k_2 = 2$, i.e. $n_1 = n_2 = 1, n_3 = 3$. So the parabolic P consists of matrices of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}.$$

As above, we see that each N-orbit contains an element $X \in \mathfrak{sl}(3,\mathbb{C})$ with $c = e = 0$. Moreover $b = 0$ because X must lie in $[\mathfrak{p}, \mathfrak{p}]^\circ$ (after suitable dual identifications), so again we get diagonal X. The remaining $[P, P]$ action is that of $SL(2,\mathbb{C}) \times \{1\}$, and the hyperkähler reduction is, as in Example 8.4, the non-regular semisimple orbit $SL(3,\mathbb{C})/S(GL(2,\mathbb{C}) \times GL(1,\mathbb{C}))$—the closed stratum of the Kostant variety for $(a, a-2a)$.

The level $(a, a, -2a)$ is also compatible with the stratum with $k_0 = k_1 = k_2 = 1$, and hence $n_1 = 1, n_2 = 2, n_3 = 3$: this is the open stratum. Now we can make $c = e = 0$ but b may be non-zero, and the residual freedom in N consists of block-diagonal matrices in N of block size 2 and 1. This will be the open stratum of the Kostant variety for $(a, a, -2a)$, and of course is not semi-simple.

Finally, let us consider the level $(0, 0, 0)$. This is compatible (using Remark 7.9) with three strata: (i) the open stratum with $k_0 = k_1 = k_2 = 1$; (ii) the stratum with $k_0 = 0, k_1 = 1, k_2 = 2$; (iii) the closed stratum with $k_0 = k_1 = 0, k_2 = 3$ corresponding to the point quiver.

On torus reduction, we will obtain the corresponding strata of the nilpotent variety: respectively, these are the regular stratum (minimum polynomial x^3), the subregular stratum (minimum polynomial x^2), and the zero element, i.e. the semisimple stratum. $\diamondsuit$

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(Dancer) Jesus College, Oxford, OX1 3DW, United Kingdom
E-mail address: `dancer@maths.ox.ac.uk`

(Kirwan) Balliol College, Oxford, OX1 3BJ, United Kingdom
E-mail address: `kirwan@maths.ox.ac.uk`

(Swann) Department of Mathematics, Aarhus University, Ny Munkegade 118, Bldg 1530, DK-8000 Aarhus C, Denmark, *and*, CP^3-Origins, Centre of Excellence for Cosmology and Particle Physics Phenomenology, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark
E-mail address: `swann@imf.au.dk`